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Dynamics and Universality of Unimodal Mappings with Infinite Criticality

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Abstract

We consider infinitely renormalizable unimodal mappings with topological type which is periodic under renormalization. We study the limiting behavior of fixed points of the renormalization operator as the order of the critical point increases to infinity. It is shown that a limiting dynamics exists, with a critical point that is flat, but still having a well-behaved analytic continuation to a neighborhood of the real interval pinched at the critical point. We study the dynamics of limiting maps and prove their rigidity. In particular, the sequence of fixed points of renormalization for finite criticalities converges, uniformly on the real domain, to a mapping of the limiting type.

1 Introduction

1.1 Overview of the problem.

Universality for unimodal mappings was discovered by Feigenbaum [14], [15] and Coullet-Tresser [7] in the case of period doubling, initially purely on the basis of

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numerical observation. For our purposes, the problem can be stated as follows. We consider mappings $H : [0,1] \to [0,1]$ in the form

$$H(x) = |E(x)|^{\ell}$$

where $\ell > 1$ is a real number and E is a smooth mapping with strictly negative derivative on $[0,1]$ which maps 0 to 1 and 1 to a point inside $(-1,0)$. Then H is unimodal with the minimum at some $x_0 = E^{-1}(0) \in (0,1)$ and x_0 is the critical point of order ℓ. The celebrated *Feigenbaum functional equation* is

$$\tau H^2(x) = H(\tau x) \tag{1}$$

for $x \in [0, \tau^{-1}]$. The equation needs to be solved for H and then necessarily $\tau^{-1} = H(1) = H^2(0)$.

The original discovery was that the the solution to Feigenbaum's functional equation can be found by iterating the following *renormalization operator*

$$\mathcal{R}(H)(x) := \frac{H^2(H(1) \cdot x)}{H(1)}$$

which can be seen as a step in the method of successive approximations for solving Equation (1). Note that $\mathcal{R}(H)$ satisfies conditions imposed in the preceding paragraph provided that $H(1) < x_0$ and then $\mathcal{R}$ can be applied again. Universality means that as soon as $\mathcal{R}^n(H)$ remains in the class described above for all n, this sequence will converge to a limit H_ℓ also in the same class, and H_ℓ is a solution to Feigenbaum's functional equation. Moreover, this limit is independent of the initial guess H, except for the rank of criticality ℓ.

The early thrust of the theory was toward actually solving Feigenbaum's equation and finding constants τ_ℓ for small values of ℓ. Next, rigorous computer-assisted proofs were developed, see [17], [18], [5]. Later, the problem was generalized to include versions of Equation (1) which involve a higher iterate of H replacing the second one.

At that point the need for a more theoretical approach to the problem became obvious. First, one could not re-run computer estimates in all infinitely many cases to which the theory seemed to apply. Secondly, while computer-assisted proofs showed the emergence of universal constants and functions, it still did not explain qualitative reasons of the phenomena. The program for solving renormalization conjectures purely with tools of dynamical systems theory was formulated by D. Sullivan in the mid-1980s. Its salient feature was strong reliance on *complex* dynamics of analytic continuations of real maps. This approach took some time to develop, but has been highly successful in the end, see [25], [23], [21], [24]. In particular, for each ℓ which is an even integer the existence of a solution H_ℓ to Equation (1), unique for the order of criticality ℓ, has been rigorously established.

This paper is concerned with the case when ℓ increases to ∞. Originally, interest in this problem came from mathematical physics literature, see [11], [28], [27], [1]. One motivation came from the expectation that the problem could shed light on other, more complicated, limit problems of statistical and quantum physics. Another reason was the obvious computational challenge of working with Equation (1) for large ℓ. With such ℓ, the renormalization operator cannot be iterated for very long because of finite accuracy and hence different procedures were needed for solving the equation. Papers quoted here all successfully dealt with this challenge obtaining consistent estimates for $\lim_{\ell\to\infty} \tau_\ell \approx 30$, for example. Their methods were cast in varying language, but were all based on the fact that functions H_ℓ for $\ell \to \infty$ approach the Fatou coordinate of a certain parabolic fixed point. In addition to developing a numerical approach, paper [11] contained a rigorous computer-assisted proof of the existence of a limiting function H_∞ which solved Equation (1) and was the limit of fixed-point transformations H_ℓ for finite ℓ,

$$H_\ell(x) = |E_\ell(x)|^\ell \ .$$

It is actually curious that such a limit may exist at all. Here, it happens because $E_\ell(x)$ for some fixed $x \neq x_0$ will tend to -1 or 1 at a rate proportional to ℓ^{-1}.

The second source of interest was the study of metric attractors of real and complex maps. In [2], the first example of an exotic attractor was shown for a unimodal map. The key estimate of the paper was obtained by adjusting the order of the critical point to a sufficiently high value. That work was followed by a program of S. Van Strien and T. Nowicki for showing the existence of a similar attractor for a complex polynomial, which would imply that the Julia set of such a polynomial has positive measure. While that program has not been followed to a successful completion, partial progress was based on choosing sufficiently high criticality and studying limits when it tended to ∞, see [29].

Contribution of this paper. We provide an analytic, not computer-assisted, proof of the existence and uniqueness of the solution to the generalized Equation (1) in a so-called EW-class of mappings with an infinitely flat critical point, see next subsection. Maps from the EW-class cover all topological equivalence classes which contain infinitely renormalizable transformations from the quadratic family and are periodic under the renormalization. Since our class contains limits of sequences H_ℓ as $\ell \to \infty$ of fixed points of renormalization for finite ℓ, the uniqueness means that H_ℓ actually converge to that limiting infinitely flat dynamics. Ultimately the EW-class is precisely the class of the limiting maps: for every order type $\aleph$, as defined in detail further, the EW-class contains one and only one map $H_\aleph$ of this type, and, moreover, $H_\aleph$ is the limit of any sequence of fixed-point maps H_ℓ of the same type $\aleph$, as the criticality ℓ tends to infinity along real numbers.

We also study basic properties of the limiting map as complex dynamical system.

Technically, our approach is based on the rigidity of towers in the sense of [23]. The class of complex maps we are working with is quite different from polynomial-like mappings studied for finite even integer ℓ. Moreover, the sequence of maps H_ℓ is not generated by any identifiable operator in a functional space. In spite of these significant differences with the standard setting, the basic approach still works. It looks likely that it should also work for other types of dynamics such as circle homeomorphisms or Fibonacci induced maps.

Main results of the paper are contained in Theorems 1- 2.

1.2 Statement of main results.

We will say that two finite sequences $(u_i)_{i=1}^{p}$ and $(u'_i)_{i=1}^{p'}$ have the same *order type* provided that $p = p'$ and $u_i < u_j$ iff $\varepsilon u'_i < \varepsilon u'_j$ for all $i, j = 1, \cdots, p$ and a fixed constant ε. The order type is an equivalence class of this relation, typically denoted with a Hebrew letter, and then $|\aleph|$ will mean the length of a sequence in $\aleph$. We will consider infinitely renormalizable maps with periodic combinatorics given by some order type $\aleph$. This means that for every n there is a restrictive interval of period $|\aleph|^n$ and the order type of points $x_0, f^{|\aleph|^{n-1}}(x_0), \cdots, f^{(|\aleph|-1)|\aleph|^{n-1}}(x_0)$ is $\aleph$. Here, x_0 is the critical point of the univalent map f.

Unimodal maps will be denoted by H, often with a subscript indicating the order of the critical point. That is, H_ℓ is assumed to be in the following form: $H_\ell(x) = |E_\ell(x)|^\ell$, where $E_\ell : [0, 1] \to \mathbb{R}$ is a C^2-diffeomorphism onto its image. Unimodal maps are normalized so that $H([0, 1]) = [0, 1]$, $H(0) = 1$ and the global strict minimum 0 is attained in $(0, 1)$. They are further assumed to be infinitely renormalizable with some combinatorial order type $\aleph$ and to satisfy the fixed point equation:

$$\tau H^{|\aleph|}(x) = H(\tau x) \; . \tag{2}$$

with $\tau > 0$. By renormalization theory, see [25], a fixed point H_ℓ for any $\ell > 1$ can be represented as $|E_\ell|^\ell$ with E_ℓ which is a diffeomorphism in the Epstein class:

Definition 1.1 *A diffeomorphism E of a real interval T' onto another real interval T is said to be in the Epstein class if the inverse map $E^{-1} : T \to T'$ extends to a univalent map $E^{-1} : (\mathbb{C} \setminus \mathbb{R}) \cup T' \to (\mathbb{C} \setminus \mathbb{R}) \cup T$.*

Our first main result is the following.

Theorem 1 *Let us fix an order type $\aleph$ and consider a sequence H_{ℓ_m}, with ℓ_m real, of unimodal maps which are infinitely renormalizable with periodic combinatorics of type $\aleph$ and satisfy the fixed point equation (2), each with its own scaling constant $\tau_m > 1$.*

If $\lim_{m\to\infty} \ell_m = \infty$*, then* H_{ℓ_m} *converge as* $m \to \infty$*, uniformly on* $[0,1]$*, to a unimodal function* H*. Also,* $\lim_{m\to\infty} \tau_m = \tau > 1$ *exists, and* H, τ *satisfy the fixed point equation (2).*

Eckmann-Wittwer class. One can say more about the analytic continuation of H. Not only does the analytic continuation provide more information about the limit, but is also crucial for our proof which relies on holomorphic dynamics. Different from the theory for finite ℓ in which the analytic continuations of limits belong to the well-known class of polynomial-like mappings, H belongs to a limiting class of mappings with a flat critical point.

Definition 1.2 *Let* H *be a smooth unimodal map defined from the interval* $[0,1]$ *into itself, with the minimum at some point* $x_0 \in (0,1)$*. Suppose that it is normalized so that* $H(x_0) = 0, H(0) = 1$ *and the orbit* $x_0, \cdots, H^{p-1}(0)$ *has order type* $\aleph$*. Then we will say that* H *belongs to the* Eckmann-Wittwer *class, EW-class for short, with combinatorial type* $\aleph$*, provided that the following conditions hold.*

1. $\tau H^p(x) = H(\tau x)$ *for some scaling constant* $\tau > 1$ *and every* $0 \le x \le \tau^{-1}$*.*
2. H *has analytic continuation to the union of two topological disks* U_- *and* U_+ *and this analytic continuation will also be denoted with* H*.*
3. *For some* $R > 1$*,* H *restricted to either* U_+ *or* U_- *is a covering (unbranched) of the punctured disk* $V := D(0,R) \setminus \{0\}$ *and* $\overline{U_+ \cup U_-} \subset D(0,R)$*.*
4. $$H(z) = \lim_{m\to\infty} \left((E_{\ell_m}(z))^2\right)^{\ell_m/2} ,$$
   where $\ell_m \to \infty$*, for each* m *the map* E_{ℓ_m} *is a diffeomorphism in the Epstein class, normalized so that* $E_{\ell_m}(0) = 1$ *and* $E_{\ell_m}(1) \in (-1,0)$*. It is understood that* $w^{\ell_m/2}$ *is the principal branch defined on the plane slit along the negative half-line and that for every compact subset* K *of* $U_+ \cup U_-$*, the right hand side of the equality is well defined on* K *for almost all* m *with uniform convergence on* K*.*
5. U_- *contains the interval* $[b_0', x_0)$ *and* U_+ *contains the interval* $(x_0, b_0]$ *where* $b_0' < 0$*,* $1 < b_0 < R$*,* $H(b_0) = H(b_0') = b_0$ *and* $H'(b_0) > 1$*.*
6. $U_\pm$ *are both symmetric with respect to the real axis and their closures intersect exactly at* x_0*.*
7. *The mapping* $G(x) := H^{p-1}(\tau^{-1}x)$ *fixes* x_0 *and* G^2 *has the following power series expansion at* x_0*:*
   $$G^2(x) = x - \epsilon(x - x_0)^3 + O(|x - x_0|^4)$$
   with $\epsilon > 0$*.*

Theorem 2 *For every sequence H_{ℓ_m} as described in the hypothesis of Theorem 1, the limiting function H belongs to the Eckmann-Wittwer class.*

The dynamics of maps in the EW-class is studied in this paper starting from Section 3.

In particular, we introduce the Julia set of EW-class maps.

Our last result is a straightening theorem for the EW-class.

As it follows from the Straightening Theorem for polynomial-like maps [DH], any map $H_{\ell,\aleph}$, if ℓ is an even integer, is quasi-conformally conjugate to a polynomial $z \mapsto z^\ell + c_{\ell,\aleph}$ in neighborhoods of their Julia sets. Here we prove that limit maps $H_\aleph$ are quasi-conformally conjugate to maps of the form $f(z) = \exp(-c(z-a)^{-2})$.

Theorem 3 *For every map $H : U_- \cup U_+ \to V$ of the EW-class there exists a map of the form $f(z) = \exp(-c(z-a)^{-2})$ with some real $a, c > 0$, such that H and f are hybrid equivalent, i.e. there exists a quasi-conformal homeomorphism of the plane h, such that*

$$h \circ H = f \circ h$$

on $U_- \cup U_+$ and $\partial h/\partial \bar{z} = 0$ a.e. on the Julia set of H. Moreover, h maps the Julia set of H onto the Julia set of f.

See last Section for the proof and comments.

1.3 Plan of the proof.

Theorems 1 and 2 follow immediately from the following two statements.

Theorem 4 *Consider a sequence of fixed-point maps H_{ℓ_m} with scaling constants τ_m, all of combinatorial type $\aleph$ and satisfying the hypothesis of Theorem 1. Let x_m denote the critical point of H_{ℓ_m}.*

Then, there is a subsequence m_p such that $x_{m_p} \to x_0$, $\tau_{m_p} \to \tau$ and $H_{\ell_{m_p}} \to H$, where H belongs to the EW-class with combinatorial type $\aleph$, critical point at x_0 and the scaling constant τ. The convergence to H is uniform on the interval $[0,1]$.

Theorem 5 *Let H_1 and H_2 be two maps belonging to the EW-class with the same combinatorial type $\aleph$. Then $H_1 = H_2$.*

Theorem 4 follows from compactness of the family $\{H_{\ell_m}\}$, which in turn follows from real and complex bounds. Further examination of limit maps shows that they belong to the EW-class.

To prove Theorem 5 we follow the strategy of [25] as realized in [23], despite of the fact that all the basic “starting conditions” of this approach break down in a transparent way for limit maps in the EW-class. For example, if H belongs to the EW-class, then:

- as a real map, H has a *flat* critical point (as we will presently argue) and many techniques do not apply, not even a "no wandering interval theorem" can be taken for granted;
- most strikingly, in spite of bounded combinatorics, the geometry of the post-critical set of H is not bounded, and, therefore, known methods of constructing quasi-conformal conjugacies do not work;
- as a complex map, H is *not* extended holomorphically through a neighborhood of its critical point; in particular, neither Fatou-Julia-Baker theory for meromorphic maps nor Sullivan-Douady-Hubbard theory [26], [10] of polynomial-like maps is applicable.

Nevertheless, the proof [23] can be adapted. We consider a *tower* generated by H, prove that it has needed chaotic properties, and derive the rigidity of the tower by showing that it cannot support an invariant line-field.

In the sequel, the combinatorics $\aleph$ is fixed, and we omit sometimes the index $\aleph$. Also, p will be used to denote the cardinality of $\aleph$.

A further comment on the EW-class. EW-class plays a role in the proof which is somewhat analogous to the impact of polynomial-like mappings in the standard theory. Both classes share a fundamental "expansion" characteristic: namely a smaller domain provides a covering of a larger one with the critical value removed. However, the critical point in the EW-class is not in the domain of analyticity.

Assume now that H belongs to the EW-class. By the functional equation (2),

$$\tau^{-1}H(z) = H(G(z))$$

which initially holds for $z \in [0,1]$, but extends to $U_- \cup U_+$ by analytic continuation. If h denotes the lifting of H to the universal cover of the disk $D_*(0,R)$ by exp, then we obtain *Abel's functional equation*

$$h(G(z)) = h(z) - \log\tau$$

which allows one to interpret h as the Fatou coordinate and $U_\pm$ as the petals of G at x_0. It also shows the nature of the singularity of H at x_0. Since the Fatou coordinate is $\log H = C_0(z-x_0)^{-2} + C_1(z-x_0)^{-1} + C_2\log(z-x_0) + O(1)$, $C_0 < 0$, we get

$$H(z) = (z-x_0)^{C_2}\exp(\frac{C_0}{(z-x_0)^2} + \frac{C_1}{z-x_0})\exp(\phi(z))$$

where $\phi(z)$ is holomorphic. The flat exponential factor precludes H from being analytic at x_0.

2 Limits as $\ell_m \to \infty$

In this section we prove Theorem 4.

2.1 Bounds

Real bounds. For all results of this section, we assume that a unimodal mapping $H_\ell(x) = |E_\ell(x)|^\ell$ is given, infinitely renormalizable with a periodic combinatorial pattern $\aleph$, and satisfying the functional equation (2) with some scaling factor τ_ℓ.

Proposition 1 *For every combinatorial pattern $\aleph$ there exist two constants $1 < T_1 < T_2 < \infty$, such that $T_1 < \tau_\ell < T_2$, for all H_ℓ.*

The proof is contained in the following two lemmas 2.2, 2.3.

First, let's make the following comment. Given a solution $H_\ell(x) = |E_\ell(x)|^\ell$ of the equation (2) with the constant $\tau = \tau_m$, let's introduce a map $g(x) = E_\ell(|x|^\ell)$. Then $g(0) = 1$, g is an even map, and 0 is the critical point of the unimodal map $g : [-1, 1] \to [-1, 1]$. It satisfies the fixed-point equation

$$\alpha g^{|\aleph|}(x) = g(\alpha x) \ , \tag{3}$$

where $\alpha = \alpha_\ell$ is a constant, which is either $+\sqrt[\ell]{\tau}$ or $-\sqrt[\ell]{\tau}$. Vice versa, to every solution $g(x) = E_\ell(|x|^\ell)$, of the equation (3), where E_ℓ is a diffeomorphism, there corresponds a solution $H_\ell(x) = |E_\ell(x)|^\ell$ of (2) with $\tau = |\alpha|^\ell$. One should have in mind the following identity between H and first return maps of g near the critical value $g(0) = 1$ of g:

Lemma 2.1 *For every $n \geq 0$,*

$$H_\ell(x) = \Lambda_n^{-1} \circ g^{|\aleph|^n} \circ \Lambda_n(x) \ , \tag{4}$$

where $\Lambda_n(x) = E_\ell(\tau^{-n}x)$ is a diffeomorphism of $[0, 1]$ onto its image.

Proof. For $x \in [0, 1]$, one can write: $\Lambda_n^{-1} \circ g^{|\aleph|^n} \circ \Lambda_n(x) = \Lambda_n^{-1} \circ g \circ g^{|\aleph|^n - 1} \circ \Lambda_n(x) = \Lambda_n^{-1} \circ g \circ g^{|\aleph|^n - 1} \circ g(|\alpha^{-n} x^{1/\ell}|) = \Lambda_n^{-1} \circ g \circ g^{|\aleph|^n}(|\alpha^{-n} x^{1/\ell}|) = \Lambda_n^{-1} \circ g(\alpha^{-n} g(|x|^{1/\ell})) = \tau^n E_\ell^{-1} \circ E_\ell(|\alpha^{-n} g(|x|^{1/\ell})|^\ell) = |g(|x|^{1/\ell})|^\ell = H_\ell(x)$.

□

Lemma 2.2 *There exists $1 < T_1$, such that $T_1 < \tau_\ell$ for all H_ℓ.*

Proof. This follows easily from real bounds of [20]. Indeed, let U_n be the central p^n-periodic interval of g_ℓ, so that the endpoints of U_n are $u_n, -u_n$, where u_n is p^n-periodic point of g_ℓ. By the functional equation, $u_n = u_0/\alpha_\ell^n$, where $u_0 < -1$ is a fixed point of g_ℓ. Let $I \supset g_\ell(U_n)$ be the maximal interval on which $g_\ell^{p^n-1}$ is monotone. Then $g_\ell^{p^n-1}(I)$ is contained in U_{n-1}. On the other hand, by [20] (Lemma 9.1+Sect. 11), there exists a universal constant C_0, such that each component of $g_\ell^{p^n-1}(I) \setminus U_n$ has length at least $C_0|U_n|/\ell$ provided n is large enough. Therefore, $|u_0\alpha_\ell^{-n+1}/(u_0\alpha_\ell^{-n})| \geq 1 + C_0/\ell$, i.e. $|\alpha_\ell| > 1 + C_0/\ell$, and the existence of the universal T_1 follows. (Let us remark that all real bounds of [20] and their proofs hold without any changes for every unimodal map of the form $E(|x|^\ell)$ where E is a diffeomorphism of the Epstein class and $\ell > 1$ is any real number.)

□

Lemma 2.3 *For every combinatorial type $\aleph$ there exists T_2 such that for all H_ℓ with combinatorial type $\aleph$, we get $\tau_\ell < T_2$.*

Proof. Decompose $H_\ell = |E_\ell|^\ell$. Let (Z_1, Z_2) denote the maximal domain of monotonicity of H_ℓ containing 0 and 1. From [20], $|E_\ell(Z_1)| \geq \sqrt[\ell]{\sigma}$ where $\sigma > 1$ is independent of ℓ, though it might depend of $\aleph$.

A key estimate here follows Lemma 3.8 in [2] and can be stated as follows. Choose $0 < A < 1$ and let $B = H_\ell(A)$ (which is necessarily positive). Let us estimate from above the $|H'_\ell(A)|$. Consider the infinitesimal cross-ratio formed by points $T, 0, A, A + dx$ where $Z_1 \leq T < 0$ is chosen so that $E_\ell(T) = \sqrt[\ell]{\sigma}$. Since E_ℓ is in the Epstein class, the cross-ratio inequality gives

$$|E'_\ell(A)|\frac{|t-1|}{T}\frac{A}{|b-1|}\frac{|A-T|}{|b-t|} < 1$$

where we denoted $b = \sqrt[\ell]{(B)}$ and $t = E_\ell(T)$. Since $\frac{|A-T|}{|T|} > 1$ and $|H'_\ell(A)| = \ell b^{\ell-1}|E'_\ell(A)|$, we get

$$|H'_\ell(A)| < \frac{|b-t|}{|t-1|}\frac{|b-1|}{A}\frac{B}{b}\ell = \frac{B}{A}\ell|b-1|\ell\frac{|t-b|}{t}\frac{1}{\ell|t-1|}\frac{t}{b} .$$

Since $|b-1| < \log b^{-1}$, $\frac{|t-b|}{t} < \log\frac{t}{b}$ $|t-1| > \log t$, we get

$$|H'_\ell(A)| < \frac{B}{A}\log b^\ell \log\frac{t^\ell}{B}\frac{1}{\log t^\ell}\frac{t}{b} .$$

Finally recalling that $t = \sqrt[\ell]{\sigma}$ and $b^\ell = B$, we get

$$|H'_\ell(A)| < \frac{B}{A}\log B^{-1}\log\frac{\sigma}{B}\frac{1}{\log\sigma}\sqrt[\ell]{\frac{\sigma}{B}} . \tag{5}$$

When $A = 1, H_\ell(1), \cdots, H_\ell^{|\aleph|-2}(1)$ then $B = H_\ell(A)$ is at least τ_ℓ^{-1}, since $\tau_\ell^{-1} = H_\ell^{|\aleph|-1}(1)$ is the closest return of the orbit of 0 to itself. For all such A, we can thus rewrite (5) as

$$|H_\ell'(A)| < \frac{B}{A} \log \tau \ell \frac{\log \tau_\ell + 1}{\log \sigma} \sqrt[\ell]{\frac{\sigma}{B}} \; .$$

Now the functional equation implies that $|(H_\ell^{|\aleph|-1})'(1)| = 1$. Therefore, if we take the product of such estimates for all A equal to

$$1, H_\ell(1), \cdots, H_\ell^{|\aleph|-2}(1) \; ,$$

we get 1 on the left-hand side.

We obtain

$$1 < \tau_\ell^{-1} \left(\log \tau_\ell \frac{\log \tau_\ell + 1}{\log \sigma} \right)^{|\aleph|-1} \times$$
$$\times \, (\tau_\ell \sigma)^{(|\aleph|-1)/\ell} \; .$$

Since for $\ell > |\aleph|$ the right-hand side goes to 0 as τ_ℓ increases to ∞, the estimate follows for all ℓ but finitely many.

□

Complex bounds.

Proposition 2 *For every combinatorial type $\aleph$, there exist constants ℓ_0, $\lambda > 1$ and R_1, such that, for every $H_\ell = |E_\ell|^\ell$ with combinatorial type $\aleph$ which satisfies the functional equation (2) with some $\tau_\ell > 1$, as soon as $\ell \geq \ell_0$, there exists $1 < R < R_1$ as follows. The function E_ℓ extends to a map from the Epstein class defined on a neighborhood of $[0,1]$, so that function $H_\ell = |E_\ell|^\ell$ extends to a unimodal function from some interval $[R'_-, R'_+]$ onto $[0, R]$, having a fixed point $b_\ell \in (1, R)$, with the following inequalities:*

$$|R'_-| \leq |R'_+| \leq \lambda^{-1} R \; .$$

The name "complex bound" comes from the fact that since for ℓ which is an even integer $H_\ell = (E_\ell)^\ell$ with E_ℓ in the Epstein class, Proposition 2 implies that H_ℓ has a polynomial-like extension onto the domain $D(0, R)$.

Proof. Proposition 2 follows from [20]. To make the reduction, we consider the dynamics of the corresponding map $g(x) = E_\ell(|x|^\ell)$ on the level of p^n-periodic central interval U_n where n is large enough. To connect this dynamics with the map H, one can use the identity(4) rewritten in the form $E_\ell^{-1}(x) = \tau^{-n} E_\ell^{-1} \circ$

$g^{-(p^n-1)}(|\alpha|^{-n}x)$, where $g^{-(p^n-1)}$ is the branch from the interval U_n to a neighborhood of $g(0)=1$. The identity holds originally in a small neighborhood of 0. On the other hand, the right-hand side extends to a real-analytic function on an interval $[-R^{1/\ell}, R^{1/\ell}]$, where $R = |\alpha^n g^{p^n}(\tilde{u})|^\ell$ and $\tilde{u}$ is a point defined in Lemma 9.1 of [20] for the p^n-periodic central interval of g. Then we apply the latter Lemma and get the result.

□

2.2 Limit maps

Our aim is to pick a convergent subsequence from H_{ℓ_m} by some kind of compactness argument. The problem is that as $\ell_m \to \infty$, then the domains of definition as U_{ℓ_m} tend to degenerate at a limit of the critical points x_{ℓ_m}.

To deal with this phenomenon, we consider inverse branches of H_{ℓ_m} corresponding to values to the left and to the right of the point x_{ℓ_m}.

From the form and normalization of mappings H_{ℓ_m}, each of them can be represented as $|E_{\ell_m}(x)|^{\ell_m}$ with E_{ℓ_m} an Epstein diffeomorphism mapping at least onto the interval $(-\sqrt[\ell_m]{R_{\ell_m}}, \sqrt[\ell_m]{R_{\ell_m}})$ with R_{ℓ_m} chosen from Proposition 2. Further from Proposition 2 one gets that $E_{\ell_m}^{-1}(D(0, \sqrt[\ell_m]{R_{\ell_m}})) \subset D(0, \lambda^{-1}R_\ell)$. By taking a subsequence we can assume without loss of generality that $R_{\ell_m} \to R \geq \lambda > 1$. Similarly, in the light of Proposition 1, we may assume that $\tau_m \to \tau > 1$. Choosing yet another subsequence, we may assume that $x_{\ell_m} \to x_0$.

We will actually invert not H_{ℓ_m}, but its lifting h_{ℓ_m} to the universal cover of $D_*(0,R)$ by exp. This will have two real branches, one mapping onto a right neighborhood of x_{ℓ_m} and one onto a left neighborhood. Their complex extensions are

$$P^+_{\ell_m}(w) := E^{-1}_{\ell_m}(\exp(w/\ell_m)) \tag{6}$$

$$P^-_{\ell_m}(w) := E^{-1}_{\ell_m}(-\exp(w/\ell_m)) \ .$$

Both transformations are defined in $\Pi_m := \{w : \Re w < \log R_{\ell_m}\}$ and map into $D(0, \lambda R_{\ell_m})$ by Proposition 2.

By Montel's theorem we can pick a subsequence m_k, such that $P^\pm_{\ell_{m_k}}$ converge to mappings $P^\pm$ defined on $\Pi_* := \{w : \Re w < \log R\}$. Since the domains vary with m, they should be normalized for example by precomposing with a translation, which tends to 0 in the limit. This implies uniform convergence on compact subsets, with the understanding that every compact subset of Π_* belongs to Π_m for almost all m. In the sequel, we will ignore this subsequence and simply assume that $P^\pm_{\ell_m}$ converge.

Let us see that $P^\pm$ are both non-constant. Note that $P^+_{\ell_m}(0) = 0$. Moreover, by the functional equation, $H^{p+1}_{\ell_m}(x_{\ell_m}) = H^p_{\ell_m}(0) = 1/\tau_m$, and, by the combinatorics, $H_{\ell_m}(1/\tau_m) = H^p_{\ell_m}(1) \in (1/\tau_m, 1)$. Therefore, there exists a point

$a_{\ell_m} \in (\log(1/T_2), 0)$, such that $P^+_{\ell_m}(a_{\ell_m}) = 1/\tau_m \subset (1/T_2, 1/T_1)$, so that $P^+_{\ell_m}(a_{\ell_m})$ are uniformly away from zero. Similarly, one can see that any limit function of the family $\{P^-_{\ell_m}\}$ is not constant as well. The considerations are slightly different in the cases $p = 2$ and $p > 2$; for example, let $p > 2$. Then $H^2_{\ell_m}(0), H^{2+p}_{\ell_m}(0) \in (H^p_{\ell_m}(0), 1) = (1/\tau_m, 1) \subset (1/T_2, 1)$; on the other hand, $P^+_{\ell_m}(a_{\ell_m}) = 1/\tau_m$, where $a_{\ell_m} = \log(H^p_{\ell_m}(1)) \in (\log(x_{\ell_m}), 0)$; the limit maps of $(P^+_{\ell_m})$ are not constants, hence, there is $c^* < 1$ such that $H^p_{\ell_m}(1) < c^*$ for all ℓ_m; therefore, $P^-_{\ell_m}(\log(H^{1+p}_{\ell_m}(1))) = H^p_{\ell_m}(1) < c^*$ while $P^-_{\ell_m}(\log(H_{\ell_m}(1)) = 1$, and the conclusion follows.

It is also clear that $P^\pm$ are both univalent. This is because for any compact subset of Π_* and ℓ_m large enough, $P^\pm_{\ell_m}$ are univalent on this set, which is evident from their defining formulas (6).

Let us define $x_0^\pm := \lim_{x\to-\infty} P^\pm(x)$. Since $(P^+)^{-1}$ in increasing on $(x_0^+, 1]$ and $(P^-)^{-1}$ is decreasing on $[0, x_0^-)$, we must have $x_0^- \le x_0^+$. We will next show that $P^+(\Pi_*) \subset \mathcal{D}((x_0^+, R'_+), \pi/2)$ and $P^-(\Pi_*) \subset \mathcal{D}((x_0^-, R'_-), \pi/2)$. We used here notations $R'_\pm$ from the statement of Proposition 2 and for any interval I, $\mathcal{D}(I, \pi/2)$ means the Euclidean disk with I as its diameter. We will concentrate on the first inclusion. It will follow once we show that for m large enough and any $w \in \Pi$, $\exp(w/\ell_m) \in \mathcal{D}(0, \sqrt[\ell_m]{R})$, by formula (6) and since E_{ℓ_m} is in Epstein class. The inclusion follows since $|\arg(\log R - w)| < \pi/2$ and exp is conformal, so

$$\lim_{m\to\infty} \arg(\sqrt[\ell_m]{R} - \exp(w/\ell_m)) = \arg(\log R - w) .$$

Checking conditions for the Eckmann-Wittwer class. We can now define a limit mapping H which will be shown to satisfy Definition 1.2.

We set $U_\pm = P^\pm(\Pi_*)$. Then $H_{|U_\pm} := \exp \circ (P^\pm)^{-1}$. H can also be defined and equal to 0 on the interval (perhaps degenerate) $[x_0^-, x_0^+]$.

We have shown that H_{ℓ_m} converge to H uniformly on any compact subset of $(x_0^+, R'_+]$ or $[R'_-, x_0^-)$, again using notations from Proposition 2. Because the mappings H_{ℓ_m} and H are unimodal, this implies uniform convergence on compact subsets of (R'_-, R'_+).

Setting out to check the conditions of Definition 1.2, we see that the functional equation is satisfied simply by passing to the limit with m. In particular, we use the fact that since H_{ℓ_m} converge uniformly, their family is equicontinuous.

The conditions second, third and fourth are satisfied by construction.

To derive the fifth condition, observe that $H(1) < 1$ while $H(R'_+) = R > R'_+$. So, there must be a fixed point b_0 between 1 and R'_+ which is unique and repelling because H has non-positive Schwarzian derivative in the light of condition 4.

With regard to the sixth condition, the symmetry with respect to the real line follows from formulas (6). We have proved the disjointness of the closures of U_- and U_+ except perhaps if $x_0^+ = x_0^-$. So we now need to prove this equality. This will require another idea and we will in fact prove property 7 first.

The associated dynamics of G. For every m, define $G_{\ell_m}(z) = H_{\ell_m}^{p-1}(z/\tau_m)$ which is well-defined and holomorphic in a neighborhood of the point x_{ℓ_m}. The functional equation yields

$$\tau_m^{-1} H_{\ell_m} = H_{\ell_m} \circ G_{\ell_m} \tag{7}$$

on the interval $[0,1]$. Since $H_{\ell_m}(x) = 0$ implies $x = x_{\ell_m}$, the functional equation implies that x_{ℓ_m} is a fixed point of G_{ℓ_m}. Since $|H_{\ell_m}(x_0 + x)| = A|x|^{\ell_m} + o(|x|^{\ell_m})$, expanding G into the power series and substituting into (7) yields $|G'((x_0)| = \tau_m^{-1/\ell_m}$. Also, equation (7) and the fact that H_{ℓ_m} is unimodal imply that x_{ℓ_m} attracts the entire interval $[0,1]$ under the iteration of G_{ℓ_m}.

Since the fixed point equation remains valid for the limit function H, if we define $G(x) = H^{p-1}(\tau^{-1}x)$, equation (7) is also satisfied with indices ℓ_m removed. We see that $G(x_0) = x_0$ and x_0 is topologically non-repelling: $|G(x) - x_0| \leq |x - x_0|$ for every $x \in [0,1]$.

Recall now that $H^{-1}(0) = [x_0^+, x_0^-] \ni x_0$.

Lemma 2.4 $G([x_0^-, x_0^+]) = [x_0^-, x_0^+]$.

Proof. From the functional equation, since $\tau^{-1}H([x_0^-, x_0^+]) = 0$, it follows that $G([x_0^-, x_0^+]) \subset [x_0^-, x_0^+]$. If it were a proper subset however, we would have $G(x) \in [x_0^-, x_0^+]$ for some $x \notin [x_0^-, x_0^+]$, which would imply $H(x) = 0$ contrary to $[x_0^-, x_0^+] = H^{-1}(0)$.

□

Lemma 2.5 *On a neighborhood of the interval* $[0,1]$ *in the complex plane* $G(z) = H^{p-1}(z/\tau)$ *is well defined, in particular analytic.*

Proof. Denote $K = [0, \tau^{-1}]$. To show the claim of the lemma, it is enough to show that $H^n(K) \cap [x_0^-, x_0^+] = \emptyset$ for any $0 \leq n \leq p-2$. Otherwise, for some $0 \leq j \leq p-2$, $0 \in H^{j+1}(K)$. On the other hand, $K = [H(x_0), H^{p+1}(x_0)]$, hence, by the combinatorics, the intervals $H^n(K)$, $0 \leq n \leq p-2$, are pairwise disjoint, a contradiction.

□

From Lemma 2.5 we conclude that G_{ℓ_m} converge to G uniformly on a complex neighborhood of $[0,1]$ and that G restricted to $[0,1]$ is a diffeomorphism in the Epstein class, in particular $SG \leq 0$. Since $|G'_{\ell_m}(x_{\ell_m})| = \sqrt[\ell_m]{\tau_m^{-1}}$, the convergence implies $(G^2)'(x_0) = 1$. Coupled with the information that x_0 is topologically non-repelling on both sides, this implies the power-series expansion:

$$G^2(z) - x_0 = (z - x_0) + a(z - x_0)^{q+1} + O(|z - x_0|^{q+1})$$

with some $a \leq 0$ and some q even. First, we prove that $a \neq 0$, i.e. G^2 is not the identity. If $G^2(z) = z$, then, for every $x \in [0, 1]$, $H(x) = H(G^2(x)) = H(x)/\tau^2$, i.e. $H(x) = 0$ and $[x_0^-, x_0^+] = [0, 1]$, a contradiction. Thus, $a < 0$.

Now we prove that $q = 2$ considering a perturbation. There is a fixed complex neighborhood W of x_0, such that the sequence of maps $(G^2_{\ell_m})^{-1}$ are well-defined in W and converges uniformly in W to $(G^2)^{-1}$. Since each H_{ℓ_m} belongs to the Epstein class, then each $(G^2_{\ell_m})^{-1}$ extends to a univalent map of the upper (and lower) half-plane into itself. It extends also continuously on the real line, and has there exactly one fixed point, which is x_{ℓ_m} and which is repelling. Therefore, by the Wolff-Denjoy theorem, $(G^2)^{-1}$ has at most one fixed point in either half-plane, and one which is strictly attracting. Thus, for any m, $G^2_{\ell_m}$ has at most three simple fixed points on W, which implies $q = 2$ by Rouche's principle. In this way, we have proved condition 7.

Finally, we can finish the proof of condition 6 by showing that $x_0^- = x_0^+ = x_0$. Indeed, x_0^- and x_0^+ are both fixed points of G^2 by Lemma 2.4, but the local form G at x_0 and the condition $SG \leq 0$ mean that x_0 is the unique fixed point of G on $[x_0^-, x_0^+]$.

We have finished the proof of Theorem 4.

3 Dynamics of EW-maps

In this section, we will construct basic dynamical theory of EW-maps, including the construction of their Julia sets and quasiconformal equivalence.

3.1 Real dynamics

Recall that an interval is called *wandering* for a unimodal map provided that all its forward images avoid the critical point and its ω-limit set is not a periodic orbit.

Proposition 3 *If H is a mapping in the EW-class with any combinatorial pattern $\aleph$, then H has no wandering interval.*

Set $p := |\aleph|$ and let $I_0 = (b_0', b_0)$ using the notation of Definition 1.2.

We have the functional identity $H^{p^n} = G^n \circ H \circ G^{-n}$ for any n on $I_n := G^n(I_0)$. To verify the identity, act on both sides by G^n from the left and use the functional equation $H \circ G = \tau^{-1} H$ and the definition $G(x) = H^{p-1}(x\tau^{-1})$ p times.

Then G^m provides a smooth conjugacy between H^{p^m} on I_m and H on I_0. Since for either connected component C of $I_0 \setminus I_1$ intervals $C, \cdots, G^{m-1}(C)$ belong to I_0 and are pairwise disjoint, the distortion of G^m on C is bounded in terms of the total nonlinearity of G on I_0 and independently of m.

Introduce the following sets of intervals: for every $m \geq 1$, let $\{I_{m,j}\}_j$ be the collection of all connected components of the first entry map from I_{m-1} into I_m.

These intervals cover I_{m-1} except for countably many points (preimages of the endpoints of I_m). Define dynamics F on $P = \cup_{m\geq 1} \cup_j I_{m,j}$: if $x \in P_m = \cup_j I_{m,j}$, then $F(x) = H^{p^{m-1}}(x)$. Then F maps homeomorphically any $I_{m,j}$ onto another interval $I_{m,j'}$ and eventually onto I_m.

Let ρ_A denote the hyperbolic metric on an interval $A = (a, b)$, i.e.

$$\rho_A(x, y) = |\log \frac{|x-a||b-y|}{|y-a||b-x|}|$$

and denote by ρ_P the metric on P, defined so that $\rho_P(x, y) = \rho_{I_{m,j}}$ if $x, y \in I_{m,j}$ or is ∞ if no such m, j exist.

Start with following lemma.

Lemma 3.1 *There exists a constant K such that for every m, j the length of $I_{m,j}$ in $\rho_{I_{m-1}}$ is less than K.*

Proof. Because G^{-m} maps intervals $I_{m,j}$ onto $I_{0,j}$ and every $I_{0,j}$ is contained in a connected component of $I_0 \setminus I_1$, and because of uniformly bounded distortion, without loss of generality we can set $m = 0$. If a sequence j_k exists such that the lengths of I_{0,j_k} go to ∞ then, perhaps by taking a subsequence, right endpoints of I_{0,j_k} tend to b_0. But if $I_{0,j_k} = (\alpha, \beta)$ with β close to the repelling fixed point b_0, then $\alpha > H(\beta)$ since $(H(\beta), \beta)$ contains a preimage of an endpoint of I_1. Thus, the hyperbolic length of (α, β) can be bounded in terms of the eigenvalue of H at b_0.

$\square$

Supposing now that a wandering interval J exists, we observe that J must be disjoint from ∂I_m for every m. This is because the endpoints of I_m are pre-repelling fixed points of $H^{p^{m-1}}$ and every one-sided neighborhood of such a fixed point will eventually cover x_0 under the iteration of $H^{p^{m-1}}$. Then, $J \subset I_{n_0,j_0}$ for some n_0, j_0. Consider the sequence of intervals $J_k := F^k(J)$ Then $(\rho_P(J_k))_{k\geq 0}$ is an increasing sequence. Moreover, each time J_k is mapped into I_m for the first time, $\rho_P(J_k) > \lambda \rho_P(J_{k-1})$. Here λ is the expansion constant of the inclusion map $I_{m,j} \to I_m$, where $J_k \subset I_{m,j}$, with the metric $\rho_P = \rho_{I_{m,j}}$ in the domain and ρ_{I_m} in the image. Observe that λ is bounded away from 1 by Lemma 3.1.

Hence, $\rho_P(J_k)$ goes to ∞ with k. But as soon as $J_k \subset I_m$, then J_k is also wandering for H^{p^m} and so contained in $I_{m,j}$, which leads to a contradiction with Lemma 3.1.

3.2 Julia set

Recall that for any interval I and $0 < \theta < \pi$ the set $\mathcal{D}(I, \theta)$ consists of all points in $\mathbb{C}$ whose distance to the "line" I in the hyperbolic metric of $(\mathbb{C} \setminus \mathbb{R}) \cup I$ is less

than a constant. Such a set is bounded by arcs of circles which intersect $\mathbb{R}$ at the endpoints of I and θ denotes the angle formed by these arcs with $\mathbb{R}$ with the convention that $\mathcal{D}(I,\theta)$ grows with the growth of θ, see [9] and [25].

A few lemmas. We begin with couple of lemmas describing the complex dynamics of H.

Lemma 3.2 *Let H belong to the EW-class with some combinatorial type $\aleph$. For every $n = 0, \cdots$ consider real points $u_-^n < x_0 < u_+^n$ defined by $H(u_-^n) = H(u_+^n) = \tau^{-n}R$. Consider a point $z \in \mathbb{C}$ and $k = 1, \cdots$ chosen so that*

$$H^k(z) \in \mathcal{D}((u_-^n, u_+^n), \pi/2)$$

but $H^{k'}(z) \notin D(0, \tau^{-n}R)$ for all $0 < k' \leq k$.

For any such choice of z, k, n there is an inverse branch of H^k defined on $\mathcal{D}((u_-^n, u_+^n), \pi/2)$ which sends $H^k(z)$ to z.

Proof. Since the Poincaré neighborhood is simply connected, the only obstacle to constructing the inverse branch may be if the omitted value 0 is encountered. Thus suppose that for some $k' > 0$, ζ, which is an inverse branch of $H^{k-k'}$ well defined on $\mathcal{D}((u_-^n, u_+^n), \pi/2)$, maps $H^k(z)$ to $H^{k'}(z)$ and its image contains 0. It follows that $H^{k-k'}(0) \in (u_-^n, u_+^n)$ and so $H^{k-k'+1}(0) < \tau^{-n}R$. It follows that $k - k' + 1$ must be a multiple of p^n, where we denote $p := |\aleph|$. Then ζ is just a real map on the real line and $\zeta(u_-^n, u_+^n) \subset [0, \tau^{-n}R)$. But since Poincaré neighborhoods are mapped into Poincaré neighborhoods of the same angle by ζ, we get $H^{k'}(x) \in \zeta(\mathcal{D}(u_-^n, u_+^n)) \subset \mathbf{D}(0, \tau^{-n}R)$ contrary to the hypothesis of the lemma.

□

Lemma 3.3 *Let H belong to the EW-class. Define $U_{+,c}$ to be the connected component of $H^{-1}(D(0,R) \cap \{z : \Re(z) > 0\})$ which contains $U_+ \cap \mathbb{R}$. Also, specify $U_{-,c}$ analogously. For some point $z \in \mathbb{C}$ suppose that $H^k(z) \in U_{+,c} \cup U_{-,c}$ for all $k \in \mathbb{N}$ and the Euclidean distance from the forward orbit of z to the ω-limit set of 0 is 0. Then $z \in \mathbb{R}$.*

Proof. First, we observe that 0 must belong to the closure of the forward orbit of z. Indeed, by hypothesis, the orbit of z is contained in $H_{+,c} \cup H_{-c} \cup \{x_0\}$ and H restricted to this set is continuous. Then by the minimality of the ω-critical set, if the orbit of z accumulates on it somewhere, then it also accumulates at 0. As soon as 0 belongs to the ω-limit set of z, we can find a sequence of iterates k_n, perhaps not strictly increasing, such that H^{k_n+1} are first entry times of z into $D(0, \tau^{-n}R)$. Then $H^{k_n}(z)$ belong to $\mathcal{D}((u_-^n, u_+^n), \pi/2)$ by the Epstein class properties postulated in Definition 1.2. Consequently, we can consider inverse branches ζ_n constructed

by Lemma 3.2. Since the orbit of z is contained in $U_{+,c} \cup U_{-,c}$, then each ζ_n will map (u_-^n, u_+^n) into some real interval T_n and $z \in \mathcal{D}(T_n, \pi/2)$. But the lengths of T_n have to go to 0 or we could find a non-trivial interval contained in infinitely many of them. Such an interval would be wandering in contradiction to Proposition 3. It follows that the distance from z to $\mathbb{R}$ must be 0.

□

The filled-in Julia set.

Definition 3.1 *If H belong to the EW-class we define its* filled-in Julia set K_H *as follows:*

$$K_H := \{z : \forall n \geq 0\; H^n(z) \in \overline{U^+ \cup U^-}\} \;\cup_{n \geq 0} H^{-n}(\{x_0\});.$$

The disadvantage of Definition 3.1 is that K_H appears to depend on the parameter R from Definition 1.2. Also, other than the name there is a priori no connection between K_H and Julia sets of globally defined holomorphic mappings, so any theory has to be developed from scratch.

Theorem 6 *For an EW-map H, the filled-in Julia set K_H is the closure of the set of all preimages of* 0 *by iterates of H. In particular, K_H is independent of the particular choice of R in Definition 1.2 and its interior is empty.*

In the course of the proof we introduce some ideas which will be used also later on. Start by observing that $K_H \cap \mathbb{R} = [b_0', b_0]$ because of the negative Schwarzian of H. On the other hand, preimages of 0 are dense in $[b_0', b_0]$ in the light of Proposition 3. Also, x_0 is not an interior point of K_H since it lies on the boundary of the domain of definition, so once we know that the preimages of 0, hence of x_0, are dense in K_H, then K_H indeed has a vacuous interior. So we only need to prove the density of the set of preimages in K_H. This is done by considering the hyperbolic metric.

Hyperbolic metric. Let ω denote the ω-limit set of the critical point x_0 by the map $H : [0,1] \to [0,1]$; The set ω is closed and forward invariant; moreover, the set $V \setminus \omega$ is open and connected. Denote by ρ the hyperbolic metric of the domain $V \setminus \omega$.

If ρ is a metric and F a function, we will write $D_\rho F(z)$ for the expansion ratio with respect to the metric ρ, thus

$$D_\rho F(z) = |F'(z)| \frac{d\rho(F(z))}{d\rho(z)}\ .$$

By Schwarz's lemma, we have $D_\rho H(z) > 1$ for every $z \in U_+ \cup U_- \setminus H^{-1}(\omega)$. We will prove that if $z \in K_H$ and no forward image of z is real, then

$$\lim_{n\to\infty} D_\rho H^n(z) = \infty \ .$$

We will observe expansion of the hyperbolic metric based on the following fact:

Fact 3.1 *Let X and Y be hyperbolic regions and $Y \subset X$ and $z \in Y$. Let ρ_X and ρ_Y be the hyperbolic metrics of X and Y, respectively. Suppose that the hyperbolic distance in X from z to $X \setminus Y$ is no more than D. For every D there is $\lambda_0 > 1$ so that $|\iota'(z)|_H \leq \frac{1}{\lambda_0}$, where $\iota : Y \to X$ is the inclusion, and the derivative is taken with respect to the hyperbolic metrics in Y and X, respectively.*

In our case, we will set $Y := V \setminus (\omega \cup H^{-1}\omega)$ and $X = V \setminus \omega$. It follows that $D_\rho H(z) \geq \lambda_d > 1$ provided that the distance from z to $H^{-1}(\omega)$ with respect to ρ is bounded by d.

Fixing $z \in K_H$ which is not eventually mapped into $\mathbb{R}$ and based on Lemma 3.3, we distinguish two eventualities. The first is that the Euclidean distance from the forward orbit of z to ω is positive. The hyperbolic distance from $H^n(z)$ to $H^{-1}(\omega)$ is bounded uniformly in n and $D_\rho H^n(z)$ grows at a uniform exponential rate.

In the second (opposite) case, Lemma 3.3 gives us a sequence n_k such that $H^{n_k}(z) \notin U_{+,c} \cup U_{-,c}$. Now the hyperbolic distance from $z' := H^{n_k}(z)$ to $H^{-1}(\omega)$ is uniformly bounded. To see this, fix attention on the case when $z' \in U_+$. The hyperbolic metric ρ_+ on $U_+ \setminus \omega$ is bigger than ρ and U_+ can be conveniently uniformized by the map $\log H$ where the branch of the log is chosen so that the map is symmetric about the real axis. The image of U_+ is the half-plane $\{w : \Re w < \log R\}$, but $\Re \log H(z') < R'$ with fixed $R' < R$ since otherwise $H(z') \notin \overline{U_+ \cup U_-}$. Since $z' \notin U_{+,c}$, then $|\Im \log H(z')| \geq \pi/2$. The set $\log H(H^{-1}(\omega))$ is doubly periodic with periods $2\pi i$ and $\log \tau$, so evidently the hyperbolic distance from $\log H(z')$ to it is bounded.

Now suppose that $z \in K_H$ and z is not in the closure of the set of preimages of 0. This implies that no forward image of z is real, so $D_\rho H^n(z) \to \infty$ as just argued. Moreover, we have shown that for some sequence of iterates H^{n_k}, the hyperbolic distance from $H^{n_k}(z)$ to $H^{-1}(\omega)$ is uniformly bounded. By pulling back to z, we see that the hyperbolic distance from z to $\bigcup_{j=1}^{\infty} H^{-j}(\omega)$ is zero, and since ω is contained in the closure of the preimages of 0, this concludes the proof of Theorem 6.

3.3 Quasi-conformal equivalence

Let now $H : U^+ \cup U^- \to V$ and $\hat{H} : \hat{U}^+ \cup \hat{U}^- \to \hat{V}$ be two maps from the EW-class with the same combinatorial type $\aleph$. We will eventually show that $H = \hat{H}$, but as the first step, we prove they are quasi-conformally conjugate.

Proposition 4 *For every pair of maps $H, \hat{H}$, both in the EW-class with the same combinatorial type $\aleph$, there exists a quasi-conformal homeomorphism ϕ_0 of the plane, symmetric w.r.t. the real axis, and normalized so that $\phi_0(0) = 0, \phi_0(1) = 1$, which conjugates H and $\hat{H}$, i.e. $\phi_0(U_-) = \hat{U}_-$, $\phi_0(U_+) = \hat{U}_+$ and $\phi_0 \circ H(z) = \hat{H} \circ \phi_0(z)$ for every $z \in U_+ \cup U_-$.*

The proof of Proposition 5 will be obtained from the following

Proposition 5 *For every pair of maps $H, \hat{H}$, both in the EW-class with the same combinatorial type $\aleph$, there is a mapping ϕ_1 defined and continuous in $\overline{U_- \cup U_+}$, quasi-conformal in the interior, symmetric about the real axis, and which can be restricted to a quasi-symmetric orientation-preserving map of the interval $\mathbb{R} \cap (U_- \cup U_+)$. Dynamically, $\phi_1(H(z)) = \hat{H}\phi_1(z)$ for every z in the forward orbit of x_0 and $\frac{\hat{R}}{R} H(z) = \hat{H}(\phi_1(z))$ for every $z \in \partial(U_- \cup U_+)$.*

Given Proposition 4, one can apply the pull-back argument [25] to the original maps $H, \hat{H}$. Once Theorem 6 has been established, the construction becomes standard.

Presentation functions. In order to show Proposition 5, we have to find an alternative to the standard method which consists of constructing first a quasi-symmetric equivalence on the real line based on the bounded geometry of the Cantor attractor. For maps in the EW-class, however, the ω-critical set ω has no bounded geometry, because ω is invariant under the map $x \mapsto G(x)$. Instead, given H, we construct a complex box mapping $h = h_H$ with simpler dynamics (post-critically finite to be precise) so that ω is a subset of "repeller" of such map. This generalizes the idea of "presentation functions", see [19], [8], which was to realize a non-hyperbolic attractor as hyperbolic repeller.

Write $p := |\aleph|$. Recall the notation $I_1 = G(I_0)$ from the proof of Proposition 4. Introduce intervals $J_1 = (0, b_0\tau^{-1})$, $J_p := I_1$ and J_q which is the connected component of $H^{q-p}(J_p)$ which contains $H^{q-1}(J_1)$ for $1 < q < p$. Then J_q, $q = 1, \cdots, p$ are pairwise disjoint intervals which cover ω and are contained in $(0, R')$ for some $R' < b_0$. Also, $H(J_p) = J_1$. Then we may proceed to define $\mathcal{J}_1 = (0, \tau^{-1}R')$, $\mathcal{J}_p$ as the preimage of $\mathcal{J}_1$ by H inside J_p, and for $1 < q < p$, the interval $\mathcal{J}_q$ is the preimage of $\mathcal{J}_p$ by H^{p-q} inside J_q. Since we decreased the intervals, $\mathcal{J}_q$ are pairwise disjoint and contained in $(0, R')$.

Let us now define the "presentation function" Π, initially only on the union of intervals $\mathcal{J}_q$. We put $\Pi(x) = \tau x$ for $x \in \mathcal{J}_1$, $\Pi(x) = H(x)$ if $x \in \mathcal{J}_q$, $1 < q < p$, and $\Pi(x) = \tau H(x)$ if $x \in \mathcal{J}_p$. We use notation: Π_q is the restriction of Π on $\mathcal{J}_q$, $1 \leq q \leq p$.

The analytic continuations of Π, $\hat{\Pi}$. To define the analytic continuation of Π more precisely, consider the following geometrical disks: $D_1 = D(0, R')$, $D_2 = \tau^{-1}D_1$. As in the preceding paragraph, R' is less than b_0 but large enough so that $J_i \subset [0, R']$ for $i = 1, \cdots, p$. Then $\hat{D}_i$ are analogously defined disks in the phase space of $\hat{H}$. Now we consider the analytic continuation of H to the following sets. We extend the linear branch Π_1 to $U_1 := D_2$. Then Π_p is extended to the "figure eight" set U_p chosen so that H restricted to each connected component of U_p is a covering of the punctured disk centered at 0 with radius $\tau^{-1}R'$. From the limit formula for H in Definition 1.2, U_p is contained in the geometric disk with diameter $\mathcal{J}_p$. Then for $1 < q < p$ we set $U_q = H^{q-p}(U_p)$ choosing the appropriate connected component of the preimage, which contains the interval $\mathcal{J}_q$.

Observe that domains U_i do not intersect. By hypotheses of the EW-class, see Definition 1.2, sets $U_2, \cdots, U_p$ are contained in geometric disks based on the corresponding $\mathcal{J}_q$, and so are pairwise disjoint and also disjoint with $U_1 = D_2$.

Plan of the proof. Π is defined by analytic continuation to $\bigcup_{q=1}^{p} U_q$ and the same construction can be carried out for $\hat{H}$, yielding a box mapping $\hat{\Pi}$.

See Figure 1 for an illustration in the case of $p = 3$.

The partial conjugacy referred to by Proposition 5 is then obtained as the conjugacy between the box mappings Π and $\hat{\Pi}$. One might wonder how that is possible, since analytically Π is no simpler than H, having exactly the same type of singularity at x_0. The answer is that the dynamics of Π is completely different from H. In particular, 0 has become a repelling fixed point, and so Π is a post-critically finite map, making the task of constructing the conjugacy much easier, again using the pull-back method.

Preparatory estimates.

Lemma 3.4 *Suppose that g is real-analytic at 0 with the following power-series expansion:*

$$g(x) = x - \varepsilon x^3 + O(|x|^4) ,$$

with $\varepsilon > 0$.

If $a_2 < 0 < a_1$ are in the basin of attraction of 0, then there exists $K > 0$ such that for every $n \geq 0$

$$K \leq \frac{|g^n(a_2)|}{|g^n(a_1)|} \leq K^{-1} .$$

Proof. The Fatou coordinates $h_\pm$ on the right and left attracting petal of g, respectively, are $\frac{1}{2\varepsilon z^2} + O(|\log z|)$ with the leading term the same on either side of 0, see [6]. It follows that for n sufficiently large

$$\frac{1}{2}\sqrt{\frac{n}{2\varepsilon}} < |g^n(a_i)| < 2\sqrt{\frac{n}{2\varepsilon}}$$

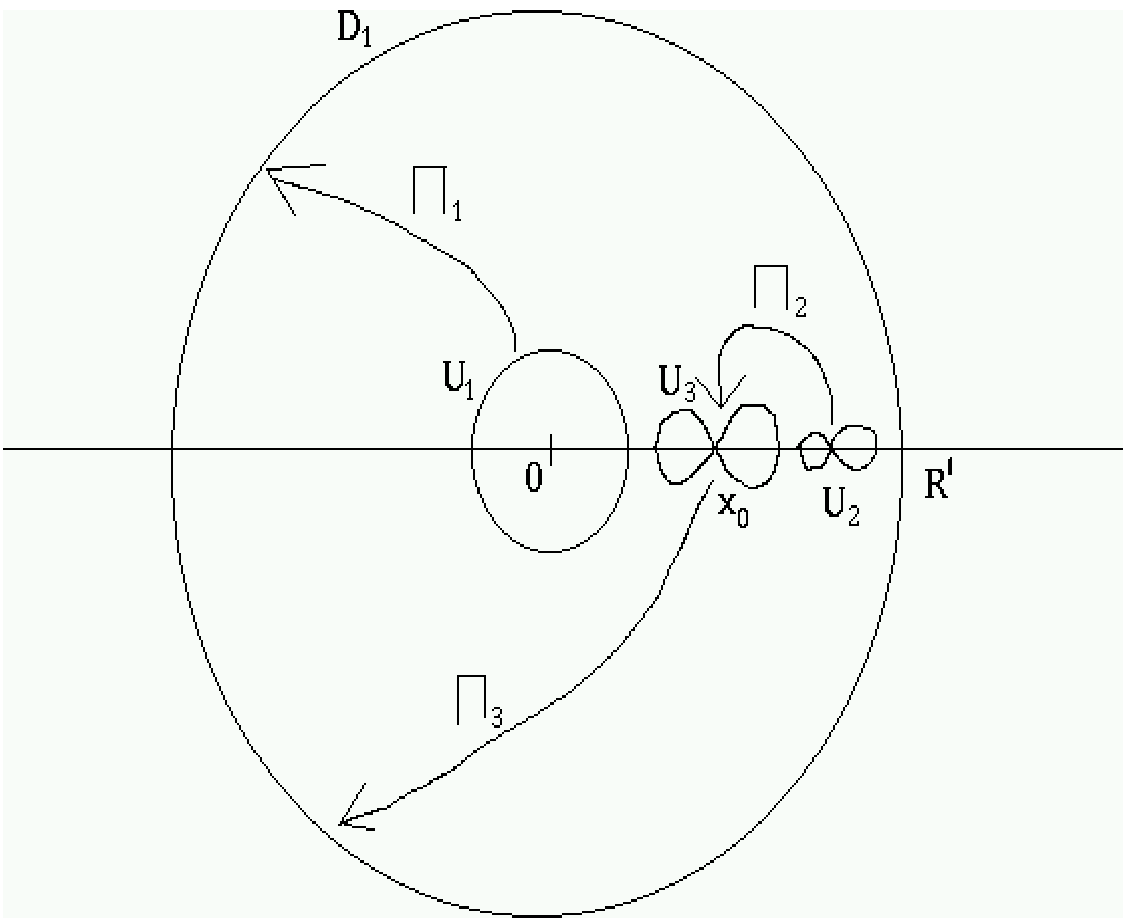

Figure 1: The box mapping Π.

and the lemma follows.

$\square$

Lemma 3.5 *Suppose now that two mappings $g, \hat{g}$ are given, both analytic in a neighborhood of* 0*, g in the same form as in Lemma 3.4 and $\hat{g}$ in the analogous form:*

$$\hat{g}(x) = x - \hat{\varepsilon}x^3 + O(|x|^4) \; ,$$

$\hat{\varepsilon} > 0$. Let $h_\pm, \hat{h}_\pm$, respectively, denote "right" and "left" Fatou coordinates symmetric about the real axis. Suppose that Υ maps 0 *to itself, is $\hat{h_+}^{-1} \circ h_+$ to the right of* 0 *and $\hat{h_-}^{-1} \circ h_-$ to the left. Then Υ is quasi-symmetric in a neighborhood of* 0*.*

Proof. We observe first that $\hat{h_+}^{-1} \circ h_+$ and $\hat{h_-}^{-1} \circ h_-$ are quasisymmetric in the respective one-sided neighborhoods of 0. This follows from the fact proved in [6] that each of the Fatou coordinates has the form $\Gamma(z^2)$ with Γ quasi-conformal on the plane, which can be normalized to a map from the left real semi-line into itself. It remains to show, see [16] Lemma 3.14, that for all $0 < \alpha < \alpha_0$, with α_0 chosen conveniently small, and fixed $K > 0$

$$K^{-1} < \frac{|\Upsilon(\alpha)|}{|\Upsilon(-\alpha)|} < K \; . \tag{8}$$

Because of the symmetry between the right and left side, we will only show the lower estimate. To this end, fix some $a_1 > 0$ and $a_2 < 0$ and set $\hat{a}_i = \Upsilon(a_i)$, $i = 1, 2$. Without loss of generality $|a_1|, |a_2| > \alpha_0$. Find the smallest n such that $g^n(a_1) \leq \alpha$. Then $g^n(a_1)/\alpha > 1/2$ if α_0 was small enough so $|g^n(a_2)| > K_1\alpha$ for some fixed $K_2 > 0$ based on Lemma 3.4. Since $\Upsilon(g^n(a_2)) = \hat{g}^n(\hat{a}_2)$ and the left branch of Υ is quasisymmetric, we get

$$|\hat{g}^n(\hat{a}_2)| > K_2|\Upsilon(-\alpha)|$$

with fixed $K_2 > 0$. But finally

$$\Upsilon(\alpha) \geq \hat{g}^n(\hat{a}_1) \geq K_3|\hat{g}^n(\hat{a}_2)|$$

with $K_3 > 0$ depending only on the choice of a_1, a_2 by Lemma 3.4. The lower estimate of inequality (8) follows.

$\square$

Construction of the partial conjugacy. We will now resume work on proving Proposition 5 first by building a partial conjugacy between Π and $\hat{\Pi}$. We start by considering the affine map $\varphi_0(z) = \frac{\hat{R}'}{R'} z$.

Next, we will construct a quasiconformal map φ_1. It is not going to be defined on the entire plane. Outside of D_1, we set $\varphi_1 = \varphi_0$. On U_1, $\varphi_1(z) = \frac{\hat{R}'}{R'} \frac{\tau}{\hat{\tau}} z$. This will ensure $\varphi_1 \circ \Pi_1 = \hat{\Pi}_1 \circ \varphi_1$ on U_1. Then on U_p we make φ_1 equal to the lift of the affine $\varphi_{1|_{U_1}}$ by H, $\hat{H}$, set up so that the lifted mapping sends $U_p \cap \mathbb{R}$ into $\mathbb{R}$ preserving the orientation. Equivalently, this is the lifting of $\varphi_{0|D_1}$ by $\Pi_p, \hat{\Pi}_p$. Finally, on each U_q, $1 < q < p$ we set $\varphi_1 = \hat{H}^{q-p} \varphi_1 H^{p-q}$ applying the appropriate inverse branch. Summarizing, φ_1 is symmetric about the real line, fixes 0, inside D_2 is defined on the union of sets $U_1, \cdots, U_p$ and satisfies $\varphi_1 \Pi = \hat{\Pi} \varphi_1$ on the union of their boundaries. What we still need is extend the domain of definition of φ_1 to the entire plane.

Before we do, observe that φ_1 restricted to the real line is quasi-symmetric provided that we interpolate on the intervals where it has not been defined, for example, by affine maps. This is clear, since on $D_1 \cap \mathbb{R}$ the map φ_1 is piecewise analytic and at the point of contact of two pieces usually it can be continued from either of them to a neighborhood of its closure. An exception occurs if the common endpoint is x_0 or one of its preimages. However, in the neighborhood of x_0 we can invoke Lemma 3.5, and the map has been propagated to the preimages of x_0 by diffeomorphic branches of $H, \hat{H}$. This allows us to construct a quasiconformal homeomorphism φ_2, of the lower half-plane onto itself, whose continuous extension matches φ_1 on the real line.

Now the reader is invited to consult Figure 2 and pay attention to the curve w marked by a thick line. This line consists of the boundary curves of domains $U_1, \cdots, U_p$ intersected with the upper half-plane, pieces of the real line between them and the boundary arcs of D_1. The key fact about w is that it is a quasi-circle. Indeed, it consists of finitely many quasi-conformally embedded arcs intersecting always with a certain angle fitting between them. In particular, at x_0 the curves are still known to posses tangent lines making angles $\pi/4$ with the real line, see [6]. Similarly, the curve $\hat{w}$ built of the analogous arcs in the phase space of $\hat{H}$, with the short-cut which is the image of the corresponding part of w by φ_0, is also a quasi-circle.

Next, we define a quasi-conformal map φ_3 of the unbounded component of the complement of w onto the unbounded component of the complement of $\hat{w}$. In the lower half-plane, we set $\varphi_3 = \varphi_2$. On $\mathbb{H}^+ \setminus D_1$ we set $\varphi_3 = \varphi_0$. On $U_i \cap \mathbb{H}^+$, $i = 1, \cdots, p$, we make $\varphi_3 = \varphi_1$. Note that we get that φ_3 extends $\varphi_{1|\mathbb{H}^+}$. But now φ_3 can be extended to the entire plane by reflecting about the quasi-circles $w, \hat{w}$. Finally, we take φ_3 from $\mathbb{H}^+$ and reflect it about the real line to $\mathbb{H}^-$, thus obtaining the desired quasi-conformal extension of the map φ_1 to the entire plane. We will still use the notation φ_1 for this extension.

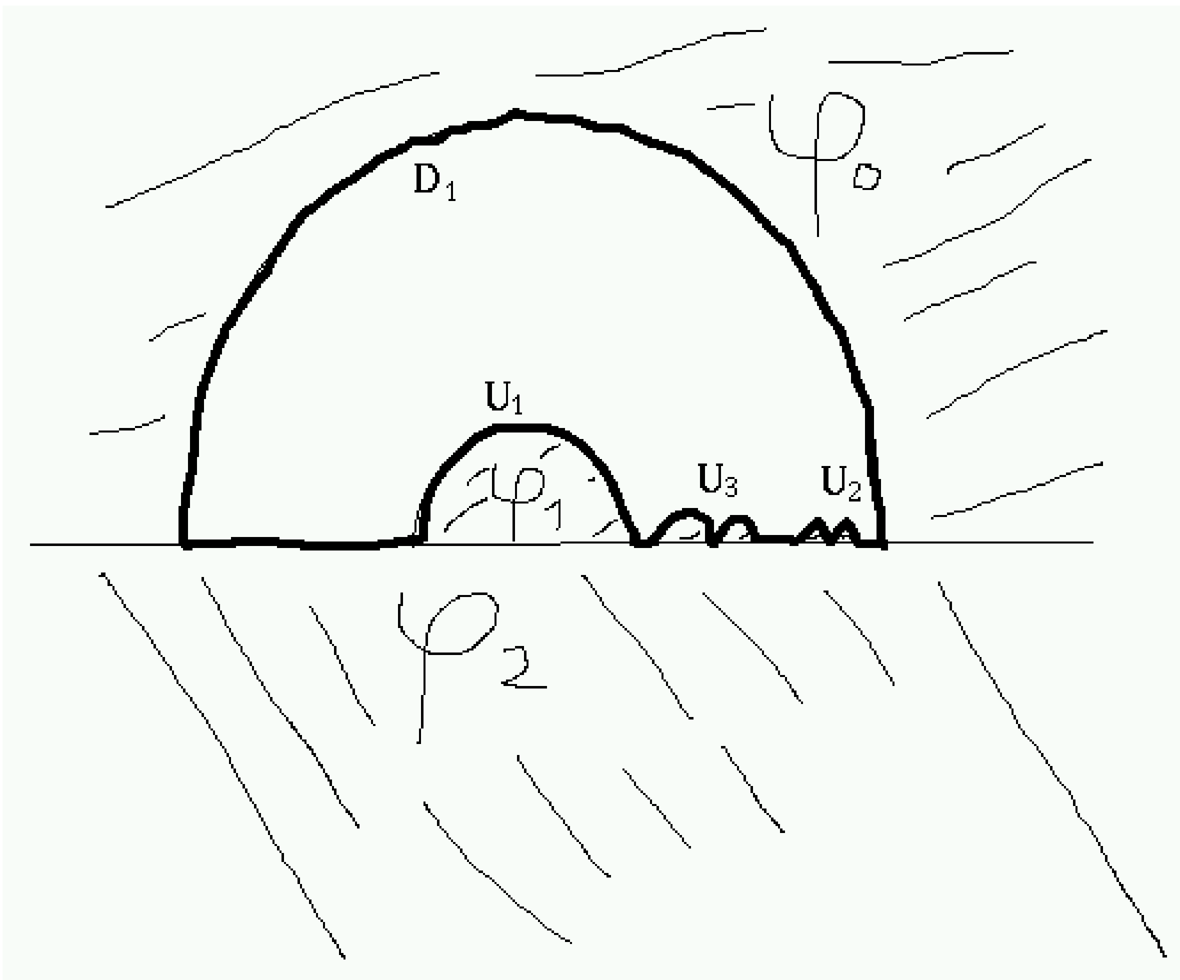

Figure 2: Construction of the partial conjugacy.

Pull-back. Now we use the standard pull-back construction. That is, we construct a sequence of quasi-conformal homeomorphisms of the plane Υ_n, $n = 0, 1, \cdots$, with $\Upsilon_0 = \varphi_1$ and Υ_n re-defined on each U_i, $i = 1, \cdots, p$ according to the formula $\hat{\Pi}^{-1} \circ \Upsilon_{n-1} \circ \Pi$. This is rigorous except for $i = p$ where $H^{-1}, \hat{H}^{-1}$ cannot be defined and one should talk instead of the lifting of Υ_{n-1} to universal covers. Since $\varphi_1(0) = 0$ and that condition is preserved by the pull-back, the lifting is well defined.

(Υ_n) form a compact family of homeomorphisms, so we can find Υ_∞ which is the limit of some subsequence of them. Then Υ_∞ is quasi-conformal and still coincides with φ_1 on ∂U_i.

Dynamics of Π. We want to show that $\Upsilon_\infty(H^n(1)) = \hat{H}^n(1)$ for all non-negative n. Since $\Upsilon_\infty(0) = 0$ as well, this will mean that Υ_∞ conjugates the forward critical orbits. The dynamics on the critical orbit under Π is simple to understand using the functional equation: 1 is periodic with period p and every image of 1 is eventually mapped to 1. Let us consider the filled-in Julia K_Π defined as the set of all points which can be forever iterated by Π. Every point $x \in K_\Pi$ has an itinerary consisting of symbols $1, \cdots, p$, where the k-th symbol being i means that $\Pi^k(x) \in U_i$. The key observation is that no two points can have the same itinerary. This follows because Π expands, though not uniformly, the hyperbolic metric of the punctured disk $V := D(0, R') \setminus \{0\}$. Indeed, the map H^{p-q+1} for $q > 1$ is a covering of V by U_q. Only on U_1 is Π an isometry. But every point is K_Π can only be iterated by Π_1 finitely many times, and it follows that the expansion ratio with respect to the hyperbolic metric along the orbit of any $x \in K_\Pi$ goes to ∞. Since all U_q, $q > 1$, have finite diameters with respect to this metric, it follows that the distance between two points with the same itinerary must be 0.

The partial conjugacy Υ_n preserves the first n symbols of any itinerary. So Υ_∞ maps K_Π into $K_{\hat{\Pi}}$ preserving the itineraries. But $H^n(1)$ and $\hat{H}^n(1)$ have the same itineraries, so it follows that $\Upsilon_\infty(H^n(1)) = \hat{H}^n(1)$.

Use of the functional equation to finish the proof. Mapping Υ_∞ satisfies the dynamical condition required on ϕ_1 in the statement of Proposition 5, but has no reason to obey the requirement imposed on the boundary of $U_- \cup U_+$. To correct this, first restrict Υ_∞ to the set U_p. Such a restriction still satisfies $\Upsilon_\infty(H^p(z)) = \hat{H}^p(\Upsilon_\infty(z))$ for every z in the forward orbit of x_0 by H^p. Additionally, by our construction, it also satisfies $\frac{\hat{R}'\tau}{\hat{\tau}R'} H(z) = \Upsilon_\infty(\hat{H}(z))$ on the boundary of U_p, which gets mapped on the geometric circle $C(0, \tau^{-1}R')$ by each branch of H. On the annulus $\{z : \tau^{-1}R' \leq |z| \leq \tau^{-1}R\}$ we can define a quasi-conformal map υ which is linear with slope $\frac{\hat{R}'\tau}{\hat{\tau}R'}$ on $C(0, \tau^{-1}R')$ and linear with slope $\frac{\hat{R}\tau}{\hat{\tau}R}$ on $C(0, \tau^{-1}R)$. Taking the appropriate lift $\hat{H}^{-1} \circ \upsilon \circ H$, we can modify Υ_∞ to a new map ϕ_1' which is defined on $U' := H^{-1}(D(0, \tau^{-1}R))$, is the same as Υ_∞, in

particular conjugating forward critical orbits of x_0 by $H^p, \hat{H}^p$, on U_p and satisfies $\frac{\hat{R}\tau}{\hat{\tau}R}H(z) = \hat{H}(\phi_1'(z))$ on the boundary of U'.

Note that ϕ_1' restricted to $U_1' \cap \mathbb{R}$ is quasi-symmetric. Indeed, ϕ_1' restricted to a smaller interval $U_p \cap \mathbb{R}$ was just a restriction of a quasi-conformal homeomorphism of the plane. We then extended it to a larger interval U' and the new mapping remains quasi-symmetric since it extends quasi-conformally to a neighborhood of each of the endpoints of U_p.

Finally, ϕ_1 as postulated by Proposition 5 is given by the formula

$$\phi_1 = \hat{G}^{-1} \circ \phi_1' \circ G \ .$$

Immediately, we see that ϕ_1 is quasi-symmetric when restricted to $U \cap \mathbb{R}$ since it is just the pulled-back of a quasi-symmetric mapping from U' by analytic maps $G, \hat{G}$.

We check the conditions starting from the functional equation $\tau^{-1}H = H \circ G$ satisfied on $U := U_- \cup U_+$. First, $G^{-1}(U') = (H \circ G)^{-1}(D(0, \tau^{-1}R)) = H^{-1}(D(0,R)) = U$ so the domain of ϕ' is $\overline{U}$ and, by an analogous argument, its range is $\overline{\hat{U}}$. For $z \in \partial U$,

$$\hat{H}(\phi_1(z)) = \hat{H} \circ \hat{G}^{-1}(\phi_1'(G(z))) = \hat{\tau}\hat{H} \circ \phi_1'(G(z)) = \hat{\tau}\frac{\hat{R}\tau}{\hat{\tau}R}H(G(z)) = \frac{\hat{R}}{R}H(z)$$

as needed.

To verify the conjugacy on the forward critical orbit, we use the identity $H^p \circ G = G \circ H$ valid at least on $[0, b_0]$, see the beginning of the proof of Proposition 3. Thus,

$$\phi_1 H^n(x_0) = \hat{G}^{-1} \circ \phi_1'(G(H^n(x_0)) = \hat{G}^{-1} \circ \phi_1'(H^{pn}(G(x_0))) =$$

$$= \hat{G}^{-1} \circ \hat{H}^{pn}(\phi_1'(G(x_0))) = \hat{H}^n(\hat{G}^{-1}(\phi_1'(G(x_0)))) = \hat{H}^n\phi_1(x_0) \ .$$

This concludes the proof of Proposition 5.

Extension of ϕ_1. The derivation of Proposition 4 from Proposition 5 is another standard application of the pull-back method. First, however, we have to extend ϕ_1 obtained from Proposition 5 to the complex plane in such a way as to make a conjugacy on the boundary of $U_- \cup U_+$. In view of the claim of Proposition 5, we simply need to extend ϕ_1 to the whole plane in such way that it becomes linear with slope $\frac{\hat{R}}{R}$ outside of $D(0,R)$, so that the main difficulty is interpolating on $D(0,R) \setminus U$.

First, we perform this interpolation on the real line, constructing a quasi-symmetric homeomorphism φ_1 which coincides with ϕ_1 on $U \cap \mathbb{R}$ and is linear with slope $\frac{\hat{R}}{R}$ outside $(-R, R)$. Next, we extend φ_1 quasi-conformally to the lower

half-plane, getting a picture similar to one shown of Figure 2. By now, we have a quasi-conformal map defined on the complement of $\mathbb{H}^+ \setminus U$. But the boundary of $\mathbb{H}^+ \setminus U$ is a quasi-circle, for the same reasons as the curve w in the proof of Proposition 5. So we can extend this to a homeomorphism of the plane by quasi-conformal reflection. Finally, we make the mapping symmetric about the real axis by reflecting from the upper half-plane into the lower. This gives the extension of ϕ_1 with the desired properties: it is a quasiconformal homeomorphism of the plane and the conjugacy condition $\phi_1(H(z)) = \hat{H}(\phi_1(z))$ now holds on the boundary of U as well as on the forward orbit of x_0.

Proof of Proposition 4. Thus, we construct a sequence of quasi-conformal homeomorphisms ϕ^n of the plane, by setting $\phi^0 = \phi_1$ and defining ϕ^n for $n > 0$ as ϕ^{n-1} outside of $U_+ \cup U_-$ and to be the lifting of ϕ^{n-1} to the universal covers $H_{|U_+}, \hat{H}_{|\hat{U}_+}$ and $H_{|U_-}, \hat{H}_{|\hat{U}_-}$. Both the lifting are uniquely defined by the requirement that ϕ^n should fix the real line with its orientation.

The sequence $\phi^n(z)$ actually stabilizes for every $z \notin K_H$. So ϕ^n converge on the complement of K_H and by taking a subsequence can be made to converge globally to some map ϕ^∞. Outside of K_H, ϕ^∞ satisfies the functional equation $\phi^\infty H = \hat{H}\phi^\infty$ and then it also satisfies it on K_H by continuity, in the light of Theorem 6. So we can set $\phi_0 := \phi^\infty$ and this concludes the proof of Proposition 4.

4 Rigidity

In this section we will prove Theorem 5 by constructing towers based on two EW-maps and showing that they must be the same.

4.1 Towers and their dynamics

Let H belong to the EW-class with some combinatorial type $\aleph$.

Definition 4.1 *Define, for $n = 0, 1, 2, \ldots$, $H_n(z) = \tau^n H(z/\tau^n)$. Then $\tau^n K_H$ is the Julia set of the map $H_n : U_n \to V_n$, where $U_n = \tau^n(U^+ \cup U^-), V_n = \tau^n V$. Note that, for any $n > m$, $H_m = H_n^{|\aleph|^{n-m}}$.*

The collection of maps $H_n : U_n \to V_n$, $n = 0, 1, \ldots$ forms the tower of H.

It is important to realize that $H_{n+1}^{|\aleph|} = H_n$ for all $n = 0, 1, \cdots$. Each H_n has its filled-in Julia set K_{H_n}, see Definition 3.1. It follows straight from the definition of H_n, that $K_{H_n} = \tau^n K_H$. Another property which follows from the definition is that the sequence K_{H_n} is increasing with n. In line with the general strategy of working with towers, we will need this:

Proposition 6 *In the tower of every EW-map H, the Julia set*

$$\bigcup_{n=1}^{\infty} \tau^n K_H$$

is dense in $\mathbb{C}$.

Dynamics in towers. *Tower dynamics* is understood as the set of all possible compositions of mappings H_i from the tower. So, if we say that z is mapped to z' by the tower dynamics, it means that a composition exists which sends z to z'. The key statement about the dynamics in towers generalizes Lemma 3.3 and uses the same notation.

Introduce the following sets. Let ω_n be the omega-limit set of 0 under the action of H_n. In particular, $\omega_0 = \omega$. Each ω_n is a closed set. Introduce $\omega_\infty = \cup_{n\geq 0}\omega_n$. It is also a closed subset of the plane. Furthermore, $\omega_\infty \cap V_n = \omega_\infty \cap U_n = \omega_n$.

Proposition 7 *For every $z \in \mathbb{C}$ which is never mapped to $\mathbb{R}$ by the tower dynamics, there exist sequences $z_n \in \mathbb{C}$ and $m_n \in \mathbb{N}\cup\{0\}$, $n = 0, 1, \cdots$, such that $z_0 = z$, z_n is an image of $H_{m_{n-1}}(z_{n-1})$ by the tower dynamics, for every $n > 0$, and at least one of the following statements is true:*

- *there exists $\eta > 0$ such that* $\operatorname{dist}(z_n, \omega_\infty) > \eta\tau^{m_n}$ *for every $n > 0$, with* dist *meaning the Euclidean distance, or*
- *for every $n > 0$*

$$\tau^{-m_n} z_n \in (U_- \cup U_+) \setminus (U_{+,c} \cup U_{-,c}) \ .$$

To prove that one of the alternative statements must hold, notice first that without loss of generality $z \notin K_h$ for any h. Otherwise, the alternative will follow by applying Lemma 3.3 inductively to the dynamics H_h.

So, assuming that z_{n-1} has been constructed we map it by the dynamics of $H_{m_{n-1}}$ until the first moment q when $w := H^q_{m_{n-1}}(z)$ is no longer in the domain of $H_{m_{n-1}}$. The only point where the set $D(0, R) \setminus (U_- \cup U_+)$ touches ω_∞ is x_0. So, if $w \notin \tau^{m_{n-1}} D(x_0, \varepsilon)$ for some $\varepsilon > 0$, then $w \in \tau^{m_{n-1}+m_0}(U_- \cup U_+)$ and $\operatorname{dist}(w, \omega_\infty) > \tau^{m_{n-1}}\eta$ with m_0 and $\eta > 0$ which depend only on ε. In that case we set $z_n := w$ and $m_n = m_{n-1} + m_0$.

Otherwise, we continue iterating $W := \tau^{-m_{n-1}}w$ by G. The connection with the tower dynamics relies on the following simple observation:

Fact 4.1 *For any q, Q, the composition $G^q(\tau^{-Q}z)$ can be represented as $\tau^{-s}\chi(z)$ where χ belongs to tower dynamics.*

Proof. If $G^{q-1}(\tau^{-Q}z) = \tau^{-s'}\chi'(z)$, then

$$G^q(\tau^{-Q}z) = G(G^{q-1}(\tau^{-Q}z)) = H^{p-1}(\tau^{-s'-1}\chi'(z)) = \tau^{-s'-1}H^{p-1}_{s'+1}(\chi'(z)) \ .$$

□

We will continue iteration by G until the first moment q' when $W' := G^{q'}(W)$ is either outside of $D(x_0, \varepsilon)$, or the distance from $\arg(W' - x_0)$ to 0 or π on the circle is less than $\pi/5$.

By specifying ε to be sufficiently small, we can achieve the following for every $u \in D(x_0, \varepsilon)$, $u \neq x_0$:

- $|\arg(G(u) - x_0) - \arg(u - x_0)| < \pi/10$,
- if the distance from $\arg(u - x_0)$ to 0 and π on the circle is less than $\pi/5$, then $u \in U_- \cup U_+$
- if $u \in U_{-,c} \cup U_{+,c}$, then the distance from $\arg(u - x_0)$ to 0 or π is less than $\pi/10$,
- $G(u)$ is in τU_-.

The first possibility is that $|W' - x_0| \geq \varepsilon$. By the properties postulated here, the distance from $\arg(W' - x_0)$ to 0 and π on the circle is at least $\pi/10$ and $W' \in \tau U_-$. By Fact 4.1, for some s we get $z_n := \tau^s W' = \chi'(w)$ for some tower iterate χ'. Then $z_n \in \tau^{s+1} U_-$ and $\operatorname{dist}(z_n, \omega_\infty) \geq \tau^s \varepsilon \sin \frac{\pi}{10}$. We set $m_n = s + 1$.

Finally, it may be that $|W' - x_0| < \varepsilon$. Then the distance from $\arg(W' - x_0)$ to $\{0, \pi\}$ on the circle is between $\pi/10$ and $\pi/5$. By the choice of ε, $W' \in (U_- \cup U_+) \setminus (U_{+,c} \cup U_{-,c})$. Again, we set $z_n = \tau^s W'$ where s comes from Fact 4.1. Setting $m_n = s$, we get

$$\tau^{-m_n} z_n \in (U_- \cup U_+) \setminus (U_{+,c} \cup U_{-,c}) \ . \tag{9}$$

This inductive construction yields a sequence of points z_n and integers m_n such that for each of them either $\operatorname{dist}(\tau^{-m_n} z_n, x_0) > \eta$ with η independent of n, as happens in the first two cases we considered, or z_n satisfies condition (9). Since one of these subsequences is infinite, Proposition 7 follows.

4.2 Expansion of the hyperbolic metric.

Hyperbolic metric. Recall that ω_n is the omega-limit set of 0 under the action of H_n, $\omega = \omega_0$, and $\omega_\infty = \cup_{n \geq 0} \omega_n$.

Let ρ_∞ be the hyperbolic metric of $\mathbb{C} \setminus \omega_\infty$. Note that ρ_∞ is invariant under the rescaling $z \mapsto \tau z$.

The following lemma is stated in terms of H, but clearly it applies to any H_k as well, because the only difference is the conjugation by a power of τ, which is the isometry of the hyperbolic metrics involved.

Lemma 4.1 *Suppose that H is an EW-map with combinatorial type $\aleph$. For any $z \in (U_- \cup U_+) \setminus H^{-1}(\omega_\infty)$, we get that the hyperbolic metric expansion ratio*

$$DH_{\rho_\infty}(z) \geq (\iota'(z))^{-1}$$

where ι is the inclusion map from $\mathbb{C} \setminus H^{-1}(\omega_\infty)$ into $\mathbb{C} \setminus \omega_\infty$ and the prime denotes its contraction ratio with respect to the corresponding hyperbolic metrics.

Proof. We can represent $H'(z) = DH_{\rho_\infty}(z)\iota'(z)$ where $H'(z)$ represents the expansion ratio of H acting from the hyperbolic metric of $\mathbb{C} \setminus H^{-1}(\omega_\infty)$ into the hyperbolic metric of $\mathbb{C} \setminus \omega_\infty$. Writing $p := |\aleph|$, we get for any $k \geq 0$ that $H = H_k^{p^k}$. Observe that $H_k^{p^k}$ is a holomorphic covering of $X_k = D(0, \tau^k R) \setminus \omega_\infty$ by $\tau^k(U_- \cup U_+) \setminus H_k^{-p^k}(\omega_\infty)$.

Hence, it is a local isometry with respect to the corresponding hyperbolic metrics. So, it is non-contracting when the hyperbolic metric of $\tau^k(U_- \cup U_+) \setminus H_k^{-p^k}(\omega_\infty)$ is replaced with the hyperbolic metric of a larger set $Y_k = \tau^k(U_- \cup U_+) \setminus H^{-1}(\omega_\infty))$.

As k tends to ∞, the hyperbolic metrics of X_k tend to $d\rho_\infty$ while the hyperbolic metrics of Y_k tend to the hyperbolic metric of $\mathbb{C} \setminus H^{-1}(\omega_\infty)$ uniformly on compact sets. It follows that $H'(z) \geq 1$ as needed.

□

Uniform expansion. Now take any point $z \in \mathbb{C}$ which is never mapped to $\mathbb{R}$ by the tower dynamics. Proposition 7 then delivers a sequence z_n. Let χ_n be the corresponding tower iterate which maps z to z_n.

Lemma 4.2 *For every D there exists $\lambda > 1$, such that for every n and every w in the ball centered at z_n with radius D with respect to ρ_∞, $D_{\rho_\infty} H_{m_n}(w) > \lambda$, provided that w is in the domain of H_{m_n}.*

Proof. By Lemma 4.1 and Fact 3.1, $DH_\rho(w) > \lambda > 1$ where λ depends only on the distance in ρ_∞ from w to $H^{-1}(\omega_\infty)$. By rescaling, the same is true for all H_k. But if either case of the alternative statement holds, points z_n are all in a uniformly bounded ρ_∞-distance from the corresponding set $H_{m_n}^{-1}(\omega_\infty)$. The same will be true for w by the triangle inequality.

□

Lemma 4.3 *For every n, let ζ_n denote the inverse branch of χ_n which maps z_n to z defined on some simply-connected set $U_n \ni z_n$. Then for every D and ε there exists n_0 such that for every $n \geq n_0$ if the diameter of U_n with respect to ρ_∞ does not exceed D, then $\zeta_n(U_n)$ is inside the hyperbolic ball of radius ε centered at z.*

Proof. Pulling back a U_n will not increase its diameter, so each time we pass z_m its radius will be shrunk by a definite factor.

□

Density of the Julia sets. We can now prove Proposition 6. For some fixed D and every n, we can find an element of $H_{m_n}^{-1}(\omega_\infty)$, moreover, a preimage of 0 by H_{m_n}, which can be joined to z_n by a simple arc γ_n of hyperbolic length which does not exceed some fixed D and which is completely contained in $\tau^{m_n}(U_- \cup U_+)$. This follows from simple geometric considerations similar to those used in the proof of Theorem 6. We can then find k which is at least equal to m_n and large enough so that the tower iterate χ_n can be represented as an iterate of H_k.

Then the inverse branch ζ_n is defined on a neighborhood of γ_n. We can apply Lemma 4.3 to get that ζ_n maps γ_n into a neighborhood of z whose diameter shrinks to 0 as n grows. Letting n go to ∞, we get that every ball centered z contains a preimage of 0 by some iterate of the tower dynamics. But every preimage of 0 in the tower belongs to some K_{H_k} and so Proposition 6 follows.

4.3 Conjugacy between towers

Given towers built for two EW-maps H and $\hat{H}$, we construct a quasiconformal conjugacy *between the towers* by rescaling the conjugacy between H and $\hat{H}$ to conjugacies $\tau^n \circ \phi_0 \circ \tau^{-n}$ of $H_n, \hat{H}_n$, pass to a limit, and get a conjugacy of the tower, which is also invariant under the rescaling:

Proposition 8 *There is a quasi-conformal homeomorphism ϕ of the plane, symmetric w.r.t. the real axis, and normalized so that $\phi(0) = 0, \phi(1) = 1, \phi(\infty) = \infty$, which conjugates every H_n with $\hat{H}_n$: $\phi \circ H_n = \hat{H}_n \circ \phi$ whenever both sides are defined. Moreover, $\phi(z) = \hat{\tau}\phi(z/\tau)$ for any $z \in \mathbf{C}$.*

The conjugacy ϕ is easily constructed based on Proposition 4. Denote $\phi^n(z) = \hat{\tau}^n \phi_0(\tau^{-n} z)$. For every n, we have

$$\phi^n H_n(z) = \hat{\tau}^n \phi_0(\tau^{-n}\tau^n H(\tau^{-n} z)) = \hat{\tau}^n \hat{H}(\phi_0(\tau^{-n} z)) = \hat{H}_n(\phi^n(z))$$

and so ϕ^n conjugates H_n to $\hat{H}_n$. Since $H_{n-1} = H_n^{|\aleph|}$, then ϕ^n also conjugates H_i to $\hat{H}_i$ for $i = 0, \cdots, n$.

Using the compactness of the family ϕ^n, we pick a limit point ϕ which conjugates the whole towers. What will require a check, however, is the invariance of ϕ under the rescaling.

Uniqueness of the conjugacy on the Julia set.

Lemma 4.4 *Suppose that H belongs to the EW-class with some combinatorics $\aleph$. Let Υ be a homeomorphism which self-conjugates H, i.e. $\Upsilon(H(z)) = H(\Upsilon(z))$ for every $z \in U_- \cup U_+$. In addition, Υ is symmetric about the real line and preserves its orientation. Then $\Upsilon(z) = z$ for every $z \in K_H$.*

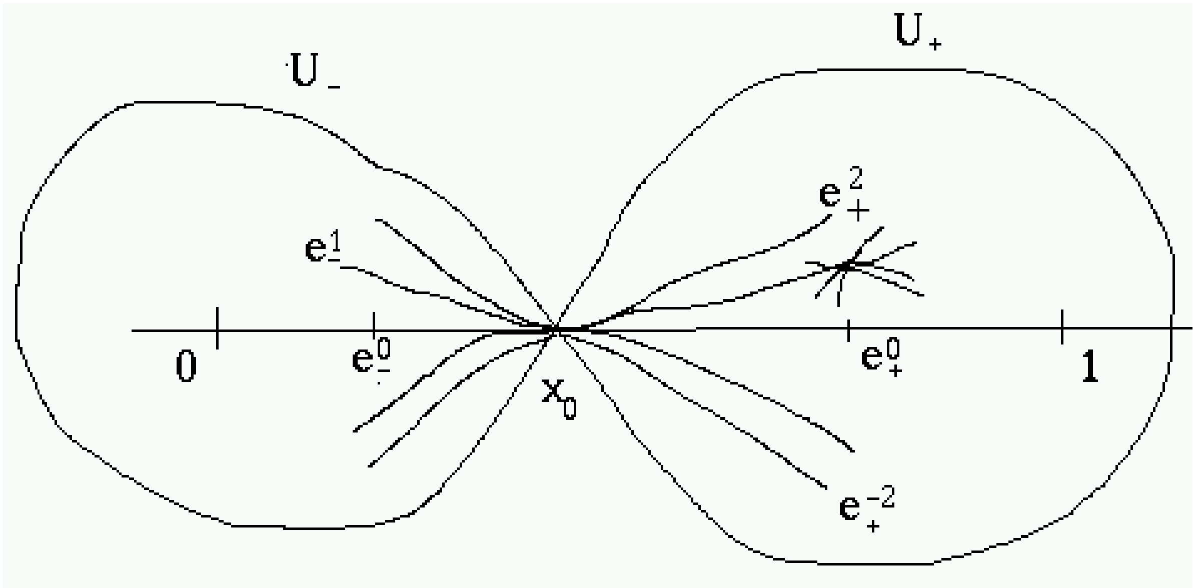

Figure 3: Edges of order 1, with some edges of order 2 branching from e^1_+.

Proof. We will consider preimages of $[0, x_0]$ by H^n and refer to them as *edges of order k*. The endpoints of each edge of order n are preimages of 0: one of order n, one of order $n+1$. Let us prove by induction that H maps each edge of order at most n onto itself, fixing the endpoints. The first non-trivial case is $n = 1$. The edges of order 1 are easy to understand: there are two infinite families of them, one in U_+ and one in U_- both branching from x_0. We can label them $(e^k_+)_{k=-\infty}^{k=+\infty}$ and $(e^k_-)_{k=-\infty}^{k=+\infty}$, respectively. See Figure 3.

Υ permutes the edges of each family. We will focus on the family e^k_- to show that this permutation is in fact the identity. Since Υ preserves the real line with its orientation, we must have $\Upsilon(e^0_-) = e^0_-$. Then if $e^1_- = \Upsilon(e^{k_1}_-)$ with $k_1 > 1$, then $\Upsilon(e^1_-)$ would have nowhere to go, since Υ must preserve the cyclic order of the edges. So $\Upsilon(e^1_-) = e^1_-$ and in this way we can inductively prove that $\Upsilon(e^k_-) = e^k_-$ for each k.

Now for an inductive step, suppose that Υ fixes all edges of order $n-1$, $n > 1$, but for some edge e of order n, $\Upsilon(e) = e' \neq e$. One endpoint of e is a preimage of 0 of order n which also belongs to an edge of order $n-1$, so it must be fixed by Υ. Thus e, e' branch out of the same point y which is the preimage of 0 of order n. Since $n > 1$, a neighborhood of y is mapped by Υ diffeomorphically on to a neighborhood of $\Upsilon(y)$. Then $\Upsilon(H(e)) = H(\Upsilon(e)) = H(e') \neq H(e)$ which is contrary to the inductive hypothesis, since $H(e)$ is already an edge of order $n-1$.

In particular, it follows that Υ fixes preimages of 0, but those are dense in K_H

by Theorem 6.

□

Coming back to the proof of Proposition 8, we observe that for any $m > n$, $\phi^n(z) = \phi^m(z)$ provided that $z \in K_{H_n}$. Indeed, booth ϕ^m and ϕ^n conjugate H_n to $\hat{H}_n$ and so $(\phi^m)^{-1} \circ \phi^n$ provides a self-conjugacy of H_n and Lemma 4.4 becomes applicable.

Now if $\phi = \lim_{k\to\infty} \phi^{n_k}$, then $\phi'(z) = \hat{\tau}\phi(\tau^{-1}z)$ is the limit of the sequence ϕ^{n_k+1}. For $z \in K_{H_m}$ and any m, the values of both sequences at z stabilize. Hence, $\phi(z) = \phi'(z)$ for any $z \in \bigcup_{m=0}^{\infty} K_{H_m}$ but this set is dense in $\mathbb{C}$ by Proposition 6. So, $\phi = \phi'$ and Proposition 8 has been demonstrated.

4.4 Invariant line-fields

We will identify *measurable line-fields* with differentials in the form $\nu(z)\frac{d\bar{z}}{dz}$ where ν is a measurable function with values on the unit circle or at the origin. A line-field is considered *holomorphic* at z_0 if for some holomorphic function ψ defined on a neighborhood of z_0, we have $\nu(z) = c\frac{\overline{\psi'(z)}}{\psi'(z)}$ for some constant c.

By a standard reasoning, Proposition 8 gives us a measurable line-field $\mu(z)\frac{d\bar{z}}{dz}$ which is invariant under the action of H_n^* for any n as well as under rescaling: $\mu(\tau z) = \mu(z)$.

We will proceed to show that μ must be trivial, i.e. 0 almost everywhere. This will be attained by a typical approach: showing first that μ cannot be non-trivial and holomorphic at any z_0 for dynamical reasons, and on the contrary, that it must be holomorphic at some point for analytic reasons and because of expansion.

Absence of line-fields holomorphic on an open set.

Lemma 4.5 *The line-field μ cannot be both holomorphic and non-trivial on any open set.*

Proof. Let μ be holomorphic in a neighborhood W. Since μ is invariant under $z \mapsto z/\tau$ and since $\cup_{n\geq 0}\tau^n K_H$ is dense in the plane, one can assume that W is a neighborhood of a point b of K_H. Moreover, since b is approximated by preimages of x_0, one can further assume that W is a neighborhood of a, such that $H^n(a) = x_0$, for some $n \geq 0$, and (shrinking W) that H^n is univalent on W. Apply H^n and see that μ is holomorphic in a neighborhood W' of x_0. Applying H one more time to $W' \cap U$, one sees that μ is holomorphic in a neighborhood of every point of a punctured disk $D(0,r) \setminus \{0\}$. Now apply the rescalings $z \mapsto \tau^n z$, $n = 0, 1, \ldots$. Hence, μ is holomorphic everywhere except for 0. In particular, μ is holomorphic around $1 = H(0)$. Since H is univalent around 0, then μ is actually holomorphic in the whole disc $D(0,r)$. Then μ cannot be holomorphic around $H^{-1}(0) = x_0$, a contradiction.

□

Construction of holomorphic line-fields. Our goal is to prove the following:

Proposition 9 *Suppose that H is a function from the EW-class which fixes an invariant line-field $\mu(z)\frac{d\bar{z}}{dz}$, which is additionally invariant under rescaling: $\mu(\tau z) = \mu(z)$. Then the line-field is holomorphic at some point. Additionally, it is non-trivial in a neighborhood of the same point unless $\mu(z)$ vanishes almost everywhere.*

Construction of holomorphic line-fields is based on the following analytic idea.

Lemma 4.6 *Consider a line-field $\nu_0\frac{d\bar{z}}{dz}$ defined on a neighborhood of some point z_0 which also is a Lebesgue (density) point for ν_0. Consider a sequence of univalent functions ψ_n defined on some disk $D(z_1, \eta_1)$ chosen so that for every n and a fixed $\rho < 1$ the set $\psi_n(D(z_1, \rho\eta_1))$ covers z_0. In addition, let $\lim_{n\to\infty} \psi_n'(z_1) = 0$. Define*

$$\mu_n(z)\frac{d\bar{z}}{dz} = \psi_n^*(\nu_0(w))\frac{d\bar{w}}{dw}$$

Then for some subsequence n_k and a univalent mapping ψ defined on $D(z_1, \eta_1)$, $\mu_{n_k}(z)$ tend to $\nu_0(z_0)\frac{\overline{\psi'(z)}}{\psi'(z)}$ on a neighborhood of z_1.

Proof. Let us normalize the objects by setting $\hat{\psi}_n := |\psi_n'(z_1)|^{-1}\psi_n$ and $\hat{\nu}_n(w) = \nu_0(|\psi_n'(z_1)|w)$. By bounded distortion, $\hat{\psi}_n(D(z_1, \rho\eta_1))$ contains some $D(z_0, r_1)$ and is contained in $D(z_0, r_2)$ with $0 < r_1 < r_2$ independent of n. By choosing a subsequence, and taking into account compactness of normalized univalent functions and the fact that z_0 was a Lebesgue point of ν_0, we can assume that $\hat{\psi}_n$ converge to a univalent function ψ and $\hat{\nu}_n$ converge to a constant line-field $\nu_0(z_0)\frac{d\bar{w}}{dw}$ almost everywhere. Since

$$\mu_n(z)\frac{d\bar{z}}{dz} = \hat{\psi}_n^*(\hat{\nu}_n(w)\frac{d\bar{w}}{dw})$$

for all n, we get

$$\mu_n(z) fracd\bar{z}dz \to \psi^*(\nu_0(z_0)\frac{d\bar{w}}{dw})$$

for $z \in D(z_1, \eta_1\rho)$ which concludes the proof of the Lemma.

□

Start with a Lebesgue point z_0 of μ. If the field is non-trivial, without loss of generality $\mu(z_0) \neq 0$. Also, we can pick z_0 so that it is never mapped on the real line and we can use Proposition 7.

We then proceed depending on which case occurs in Proposition 7. In the first case, we choose a point Z to be an accumulation point of $\tau^{-m_n}z_n$. Without loss of

generality, we suppose that $\tau^{-m_n} z_n \to Z$. The distance from Z to ω_∞ is positive and we can denote it by $2\eta_1$. Then, for any n we can find an inverse branch ζ_n of the tower iterate χ_n mapping z_0 to z_{m_n} defined on $D(z_{m_n}, \tau^{m_n}\eta)$. One easily checks that functions $\psi_n(z) = \zeta_n(\tau^{m_n} z)$ defined on $D(Z, \eta)$ satisfy the hypotheses of Lemma 4.6. In particular, their derivatives go to 0 because $D_{\rho_\infty}\chi_n(z_0)$ go to ∞ by Lemma 4.3.

To consider the second case of Proposition 7, fix attention on some n. The first observation is that without loss of generality $|H_{m_n}(z_n)| < R'\tau^{m_n}$ with some $R' < R$ independent of n. Indeed, all points on the circle $C(0, \tau^{m_n} R)$ are in distance $\eta\tau^{m_n}$ from ω_∞ for some η positive. So if this additional property fails for infinitely many n, we can reduce the situation to the first case already considered.

Now the key observation is that for every n the point z_n has a simply connected neighborhood Y_n, a point $y_n \in Y_n$ such that the distance in the hyperbolic metric of Y_n from z_n to y_n is bounded independently of n. Finally, Y_n is mapped univalently by H_{m_n} so that for some integer p_n and $\eta > 0$ which is independent of n the image covers $\tau^{p_n}(D(i, \eta))$ with $H_{m_n}(y_n) = \tau^{p_n} i$. To choose such Y_n and y_n, uniformize the component of $\tau^{m_n}(U_- \cup U_+)$ which contains z_{m_n} by the map $\Psi(z) = \log H_{m_n}(z)$ where the branch of the log is chosen to make the mapping symmetric about the real axis. Ψ maps onto the region $\{\Re w < m_n \log\tau + \log R\}$ and $\Re\Psi(z_n) < m_n \log\tau + \log R'$. In addition, $|\Im\Psi(z_n)| > \pi/2$ as the consequence of $\tau^{-m_n} z_n \notin U_{-,c} \cup U_{+,c}$. Then Y_n can be conveniently chosen in the Ψ-coordinate as a rectangle of uniformly bounded size.

Once y_n, Y_n, p_n were chosen, we easily conclude the proof. Let $R_n : D(0,1) \to Y_n$ be Riemann maps of regions Y_n with $R_n(0) = y_n$. Then we can set $\psi_n = (\chi_n)^{-1} \circ R_n$ where χ_n are maps specified in Proposition 7. Maps ψ_n satisfy the conditions of Lemma 4.6. In particular, $|R_n^{-1}(z_n)|$ is bounded independently of n as a consequence of the construction of Y_n.

From this and Proposition 7, the derivatives of ψ_n at $R_n^{-1}(z_n)$ go to 0, and then the same can be said of $\psi_n'(0)$ by bounded distortion. So, by passing to a subsequence, we get that $R_n^*(\mu(z)\frac{d\overline{z}}{dz})$ tend a.e. to a holomorphic line-field $\nu\frac{d\overline{w}}{dw}$ on a neighborhood of 0.

To finish the proof, we ignore the fact that a subsequence has been chosen and consider mappings $T_n := \tau^{-p_n} H_{m_n} \circ R_n$ defined on the unit disk. We have $T_n^*(\mu(z)\frac{d\overline{z}}{dz}) = R_n^*(\mu(z)\frac{d\overline{z}}{dz})$ for every n. Maps T_n are all univalent and have been normalized so that $T_n(0) = i$ and the image of $D(0,1)$ under T_n contains $D(i, \eta)$ for a fixed $\eta > 0$, but avoids 0. Then T_n is a compact family of univalent maps and has a univalent limit T. Then it develops that μ in a neighborhood of i is the image under T of the holomorphic line-field ν from a neighborhood of 0, hence is holomorphic.

Proof of Theorem 5. From Lemma 4.5 and Proposition 9 we conclude that any measurable line-field invariant under the tower of a EW-mapping and under

the rescaling by τ must be trivial. But as soon as the conjugacy ϕ constructed in Proposition 7 is non-holomorphic, it gives rise to a non-trivial line field with those properties. Hence, the conjugacy between any two EW-maps with the same combinatorial pattern must be holomorphic, and under our normalizations that means the identity.

This proves Theorem 5 which was the last missing link in the proof of our results.

5 The Straightening Theorem for EW-maps

We prove here

Theorem 7 *For every map $H : U_- \cup U_+ \to V$ of the EW-class there exists a map of the form $f(z) = \exp(-c(z-a)^{-2})$ with some real $a, c > 0$, such that H and f are hybrid equivalent, i.e. there exists a quasi-conformal homeomorphism of the plane h, such that*

$$h \circ H = f \circ h$$

on $U_- \cup U_+$ and $\partial h/\partial \bar{z} = 0$ a.e. on the filled-in Julia set of H.

We will see below that h maps the filled-in Julia set K_H of H onto the Julia set J_f of f.

Proof. Remind that $V = D(0, R) \setminus \{0\}$. Making a linear change of variable, one can assume that $R < 1$. Let us choose real $m > 0$, $0 < n < R$, as follows. Consider the map $p(z) = \exp(-m(z-n)^{-2})$, and the set $\Omega = p^{-1}(V)$. Then m, n are chosen so that $0 \in \Omega$ and $\overline{\Omega} \subset V \cup \{0\}$. As in the proof of Proposition 5, one can further choose a quasi-conformal homeomorphism φ of the plane, such that $\varphi : V \setminus U_- \cup U_+ \to V \setminus \Omega$ is one-to-one, and, most important, $\varphi(z) = z$ off V, and $\varphi \circ H = p \circ \varphi$ on the boundary of $U_- \cup U_+$. Also, φ is symmetric w.r.t. the real axis. Since $1 \notin V$, we have $\varphi(1) = 1$, also $\varphi(\infty) = \infty$, and one can assume that $\varphi(0) = 0$. Now define an extension of H to a map $\tilde{H} : \mathbf{C} \setminus \{x_0\} \to \mathbf{C} \setminus \{0\}$ as follows: $\tilde{H} = H$ on $U_- \cup U_+$, and $\tilde{H} = \varphi^{-1} \circ p \circ \varphi$ on $\mathbf{C} \setminus (U_- \cup U_+)$.

Fact 1. *Observe that since Ω is the full preimage of V by p, $p(z) \in \mathbf{C} \setminus V$ iff $z \in \mathbf{C} \setminus \Omega$.*

Define a complex structure σ a.e. on the plane as follows. Let σ_0 be the standard one. Then $\sigma = \varphi^*(\sigma_0)$ on $\mathbf{C} \setminus \overline{U_- \cup U_+}$; $\sigma = (H^n)^*(\sigma)$ on $H^{-n}(V \setminus \overline{U_- \cup U_+})$, $n = 0, 1, 2, ...$; $\sigma = \sigma_0$ on the rest. Note that $\sigma = \sigma_0$ off V.

As it follows from Fact 1 and since H is holomorphic, we get

Fact 2. *σ is correctly defined, $\tilde{H}$-invariant, and $||\sigma||_\infty < 1$.*

Let h be a quasi-conformal homeomorphism of the plane, such that $h_*(\sigma) = \sigma_0$, $h(0) = 0, h(1) = 1, h(\infty) = \infty$. Also, h is symmetric w.r.t. the real axis, because σ is symmetric. Denote $a = h(x_0)$. Define $f : \mathbf{C} \setminus \{a\} \to \mathbf{C} \setminus \{0\}$ by

$f = h \circ \tilde{H} \circ h^{-1}$. Then f is holomorphic because $f_*(\sigma_0) = \sigma_0$. We need to show that $f(z) = \exp(-c(z-a)^{-2})$, for some real $c > 0$. To this end, notice first that from the definition of f it follows that there exists $\lim_{z\to\infty} f(z) = h(1) = 1$, and that $f(z) \neq 0$ for every $z \in \mathbf{C} \setminus \{a\}$. Hence, the function $\tilde{f}(z) := 1/f(a + 1/z)$ is entire. Besides, $\tilde{f}(z) \neq 0$ for any z. Thus there exists another entire function u, such that $\tilde{f} = \exp(u)$, therefore,

$$f(z) = \exp(-u(1/(z-a))).$$

Let us study singular points of u^{-1} using the formula

$$u^{-1}(w) = [h \circ \tilde{H}^{-1} \circ h^{-1}(\exp(-w)) - a]^{-1}.$$

Since $w \in \mathbf{C}$, $\exp(-w) \neq 0$, hence, $h^{-1}(\exp(-w)) \neq 0$. If $\exp(-w_0) \neq 1$, then w_0 is not a singular point of u^{-1}. If $\exp(-w_0) = 1$ but $\tilde{H}^{-1} \circ h^{-1}(\exp(-w_0)) \neq \infty$, then again w_0 is not a singular point. At last, if $\tilde{H}^{-1} \circ h^{-1}(\exp(-w_0)) = \infty$, then w_0 is a singular point, because then, for w close to w_0, there are two different preimages $\tilde{H}^{-1} \circ h^{-1}(\exp(-w))$ close to ∞, which give two different preimages $u^{-1}(w)$ close to zero. Hence, $w_0 = 0$ is a singular point of u^{-1}. Now, if $w_0 = 2\pi i k$, $k \in \mathbf{Z} \setminus \{0\}$, then, from the symmetry w.r.t. the real axis and from the continuation along a path γ joining 0 and w_0, we see using the formula for u^{-1}, that the path $\tilde{H}^{-1} \circ h^{-1}(\exp(-\gamma))$ is not closed and starts at ∞, hence $\tilde{H}^{-1} \circ h^{-1}(\exp(-w_0)) \neq \infty$. Therefore, the only singular point of u^{-1} is zero, with the square-root singularity at this point, and, moreover, $u^{-1}(0) = 0$. Thus, $u(z) = cz^2$, and we are done.

$\square$

Now we can make use of the theory of [3], [13], [4] to describe some basic features of the Julia set of f. Remind that the Fatou set F_f is defined as the largest open set in which all f^n are defined, holomorphic and form a normal family, and the Julia set J_f is the complement $\hat{\mathbf{C}} \setminus F_f$.

Proposition 10 *Let $V_f = h(V)$ and $U_f = f^{-1}(V_f) = h(U_- \cup U_+)$. Then:*
(a) the preimages of the point a are dense in J_f,
(b) J_f is the closure of the set of such z which never leave U_f under the iterates.
(c) J_f is connected.
(d) the Fatou set $F_f = \hat{\mathbf{C}} \setminus J_f$ consists of one component, which is the basin of attraction of an attractive (real) fixed point of f. Finally, F_f is simply-connected on the sphere.

Proof. (a)-(d) follow from a series of observations.
(1). The set $E = E(f)$ of singularities of f consists of one point a. Hence, if $E_n = \cup_{j=0}^{n-1} f^{-j}(E) = \cup_{j=0}^{n-1} f^{-j}(a)$, then, by [3], $J_f = \overline{\cup_{n=0}^{\infty} E_n}$. This proves (a).
(2). The set $C(f)$ of singular values of f^{-1} consists of the point $f(\infty) = 1$.

(3). Hence, f belongs to the class **MSR** defined in [3]. In particular [3], f has no Baker domains as well as wandering domains.

(4). We have $b_f := h(b_0) > a$, and b_f is a repelling fixed point of f. Also, f is strictly increasing on $(a, +\infty)$ and $f(\infty) = 1$. Therefore, there exists an attracting fixed point z_0 of f, $b_f < z_0 < 1$. The iterates of the singular value tend to this fixed point z_0. Hence, for every component W of F_f, an iterate of W is the immediate basin of attraction W_0 of z_0.

(5). Since $E_n \subset U_f$ for all n, then, by (1), the domain $\hat{\mathbf{C}} \setminus \overline{U_f}$ is disjoint with J_f. It also contains z_0. Hence, $\hat{\mathbf{C}} \setminus \overline{U_f} \subset W_0$. On the other hand, $z_0 \in \mathbf{C} \setminus V_f$ and $f^{-1}(\mathbf{C} \setminus V_f) = \mathbf{C} \setminus U_f$, hence, $f^{-1}(z_0) \subset \mathbf{C} \setminus U_f \subset W_0$. Therefore, W_0 is completely invariant, and $F_f = W_0 = \cup_{n=0}^{\infty} f^{-n}(\hat{\mathbf{C}} \setminus V_f)$.

(6). By (5), $z \in F_f \cap U_f$ iff an iterate of z hits $V_f \setminus U_f$. Therefore, we have proved that $J_f = \cap_{n \geq 0} f^{-n}(\overline{U_f}) \cup \cup_{n \geq 0} f^{-n}(a)$. In particular, J_f is connected, and F_f is simply-connected.

$\square$

As a corollary, we get a new (indirect) proof of Theorem 6:

Corollary 5.1 *$K_H = h^{-1}(J_f)$, it has no interior, and the preimages of x_0 are dense in K_H.*

Vice versa, one can also gain an information about the dynamics of f from what we know already about the maps $H_\aleph$. For example, we obtain from Proposition 6 that the union of rescaled (around zero) Julia sets of f is dense in the plane. Another information concerns the map f on the real line; let's extend it to the point a continuously. Then $f : \mathbf{R} \to \mathbf{R}$ is a unimodal C^∞ map with the flat critical point at a. Since f and $H_\aleph$ are quasi-conformally conjugate, then the ω-limit set of the critical point a under the dynamics of $f : \mathbf{R} \to \mathbf{R}$ has no bounded geometry.

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